

FOURIER SERIES and Fourier Transforms

Fourier Series: 1. Fourier series is an infinite series representation of a periodic functions, in terms of the trigonometric sines & cosines.

2. Fourier series is possible for continuous functions, periodic functions & functions discontinuous in their values & derivatives.

3. Fourier series is useful to solve partial differential equations with boundary value problems.

Euler's (Fourier-Euler) Formulae

Definition: Let the function $f(x)$ be periodic of period ' 2π '. The fourier series of $f(x)$ in the interval $[c, c+2\pi]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx, \text{ where}$$

$$a_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x) dx, \quad a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx dx, \quad n=1, 2, 3, \dots$$

$$b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx dx, \quad n=1, 2, 3, \dots$$

Here, a_0, a_n, b_n are called the fourier coefficients & the above coefficients are also known as Euler's coefficients.

Note: 1) If $c=0$ then the interval becomes $[0, 2\pi]$ & the constant coefficients are given by

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx, \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx.$$

2) If $c=-\pi$ then the interval becomes $[-\pi, \pi]$ & the

constant coefficients are given by

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

Conditions for Fourier Series or Dirichlet's conditions:

* A function $f(x)$ has a Fourier series expansion of the form $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$, where a_0, a_n, b_n are constants, provided

1. $f(x)$ is periodic, single valued & finite.
2. $f(x)$ has a finite no. of discontinuities in any one period.
3. $f(x)$ has at most finite no. of maxima & minima in any given interval.

These conditions are known as Dirichlet's conditions.

Important Formulae:

1. $\sin n\pi = \sin(n+1)\pi = \sin 2n\pi = \sin 2(n+1)\pi = 0, \forall n \in \mathbb{Z}$, where

$\mathbb{Z}$ is the set of all integers.

2. $\cos n\pi = (-1)^n, \cos(n+1)\pi = (-1)^{n+1}, \cos 2n\pi = (-1)^{2n}, \cos 2(n+1)\pi = (-1)^{2(n+1)}, \forall n \in \mathbb{Z}$

3. $\sin(n+\frac{1}{2})\pi = (-1)^n, \forall n \in \mathbb{Z}$

4. $\cos(n+\frac{1}{2})\pi = 0, \forall n \in \mathbb{Z}$

1. Expand $f(x) = \left(\frac{\pi-x}{2}\right)^2, 0 < x < 2\pi$ in a Fourier series.

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ be the Fourier series

where a_0, a_n, b_n are the obtained using Euler's formulae

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) \cdot dx = \frac{1}{\pi} \int_0^{2\pi} \left(\frac{\pi-x}{2}\right)^2 dx \quad \text{ILATE (C)}$$

$$= \frac{1}{4\pi} \int_0^{2\pi} (\pi-x)^2 dx = \frac{-1}{4\pi} \int_0^{2\pi} (\pi-x)^2 \cdot (-1) dx$$

$$= \frac{-1}{4\pi} \left[\frac{(\pi-x)^3}{3} \right]_0^{2\pi}$$

$$\left(\because \int (f(x))^n \cdot f'(x) dx = \frac{(f(x))^{n+1}}{n+1} \right)$$

$$= -\frac{1}{12\pi} [-\pi^3 - \pi^3] : \frac{2\pi^3}{12\pi} = \frac{\pi^2}{6} \Rightarrow \boxed{a_0 = \frac{\pi^2}{6}}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} \left(\frac{\pi-x}{2}\right)^2 \cos nx \, dx$$

$$= \frac{1}{4\pi} \int_0^{2\pi} (\pi-x)^2 \cos nx \, dx$$

$$f(x) = (\pi-x)^2, \quad g(x) = \cos nx$$

$$f'(x) = -2(\pi-x) \quad \int g(x) \, dx = \frac{\sin nx}{n}$$

By integration by parts,

$$\int f(x) g(x) \, dx = f(x) \int g(x) \, dx - \int (f'(x) \int g(x) \, dx) \, dx$$

$$\therefore a_n = \frac{1}{4\pi} \left[\left((\pi-x)^2 \frac{\sin nx}{n} \right)_0^{2\pi} - \int_0^{2\pi} (-2(\pi-x) \frac{\sin nx}{n}) \, dx \right]$$

$$= \frac{1}{4\pi} \left[\left(\pi^2 \frac{\sin 2n\pi}{n} \right) - \pi^2 (0) \right] + \frac{1}{2n\pi} \int_0^{2\pi} (\pi-x) \sin nx \, dx$$

$$= \frac{1}{4\pi} (0-0) + \frac{1}{2n\pi} \int_0^{2\pi} (\pi-x) \sin nx \, dx = \frac{1}{2n\pi} \int_0^{2\pi} (\pi-x) \sin nx \, dx$$

$$= \frac{1}{2n\pi} \left[\left((\pi-x) \left(-\frac{\cos nx}{n} \right) \right)_0^{2\pi} - \int_0^{2\pi} (-1) \left(-\frac{\cos nx}{n} \right) \, dx \right]$$

$$= \frac{1}{2n\pi} \left[\left(\frac{1}{n} \left[(-\pi) \cos 2n\pi - \pi (1) \right] \right) - \frac{1}{2n^2\pi} \left(\frac{\sin nx}{n} \right)_0^{2\pi} \right]$$

$$= \frac{1}{2n\pi} \left[\frac{1}{n} [-\pi - \pi] \right] - \frac{1}{2n^3\pi} [\sin 2n\pi - 0]$$

$$= \frac{1}{2n\pi} \left[\frac{1}{n} [-2\pi] \right] - 0 = -\frac{1}{n^2} \Rightarrow \boxed{a_n = \frac{1}{n^2}}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_0^{2\pi} \left(\frac{\pi-x}{2}\right)^2 \sin nx \, dx$$

$$= \frac{1}{4\pi} \int_0^{2\pi} (\pi-x)^2 \sin nx \, dx = \frac{1}{4\pi} \left[\left((\pi-x)^2 \left(-\frac{\cos nx}{n} \right) \right)_0^{2\pi} - \int_0^{2\pi} -2(\pi-x) \left(-\frac{\cos nx}{n} \right) \, dx \right]$$

$$= \frac{1}{4\pi} \left[\left(\frac{1}{n} \left[\pi^2 \cos 2n\pi - \pi^2 \right] \right) - \frac{1}{2n\pi} \int_0^{2\pi} (\pi-x) \cos nx \, dx \right]$$

$$= \frac{1}{4\pi} \left[\frac{1}{n} (\pi^2 - \pi^2) \right] - \frac{1}{2n\pi} \left[\left((\pi-x) \frac{\sin nx}{n} \right)_0^{2\pi} - \int_0^{2\pi} (-1) \frac{\sin nx}{n} \, dx \right]$$

$$= 0 - \frac{1}{2n\pi} \left[\frac{1}{n} (k-\pi) \sin 2n\pi - \pi \cos \right] = \frac{1}{2n^2\pi} \frac{-\cos 2n\pi}{n} - \frac{1}{n^2} (\cos 2n\pi - \cos 0)$$

$$= \frac{1}{2n\pi} \left[0 - \frac{1}{n} (0-\pi) \right] = 0 \Rightarrow \boxed{b_n = 0}$$

Substituting the values of a_0, a_n, b_n in (1), we get

$$f\left(\left(\frac{\pi-x}{2}\right)^2\right) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx + \sum_{n=1}^{\infty} 0 \sin nx$$

$$\therefore f\left(\left(\frac{\pi-x}{2}\right)^2\right) = \frac{\pi^2}{12} + \cos x + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \dots, \text{ is the required Fourier series.}$$

Imp Q. Obtain Fourier series for the function, $f(x) = x \sin x$ in $0 < x < 2\pi$

sol. given $f(x) = x \sin x$, $0 < x < 2\pi$

The Fourier Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ (1)

where $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) \cdot dx$

$$= \frac{1}{\pi} \int_0^{2\pi} x \sin x \, dx$$

$$= \frac{1}{\pi} \left[(x \cdot (-\cos x)) \Big|_0^{2\pi} - \int_0^{2\pi} (-\cos x) \, dx \right]$$

$$= \frac{1}{\pi} \left[(2\pi \cos 2\pi - 0) + (\sin x) \Big|_0^{2\pi} \right]$$

$$= \frac{1}{\pi} \left[-2\pi + 0 \right] + \frac{1}{\pi} (0 - 0) = -2 \Rightarrow \boxed{a_0 = -2}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \int_0^{2\pi} x \sin x \cos nx \, dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x (2 \cos nx \sin x) \, dx = \frac{1}{2\pi} \int_0^{2\pi} x [\sin(n+1)x - \sin(n-1)x] \, dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x \sin(n+1)x \, dx - \frac{1}{2\pi} \int_0^{2\pi} x \sin(n-1)x \, dx$$

$$a_n = \frac{I_1}{2} - \frac{I_2}{2} \quad \text{--- (2)}$$

To find I_1 , I_2 $I_1 = \frac{1}{2\pi} \int_0^{2\pi} x \sin(n+1)x \, dx = \frac{1}{2\pi} \left[\left(x \cdot \left(\frac{-\cos(n+1)x}{n+1} \right) \right) \Big|_0^{2\pi} - \int_0^{2\pi} \left(\frac{-\cos(n+1)x}{n+1} \right) \, dx \right]$

$$= \frac{1}{2\pi} \left[\frac{-1}{n+1} (2\pi \cos 2(n+1)\pi - 0) \right] + \frac{1}{2\pi(n+1)} \left(\frac{\sin(n+1)x}{n+1} \right)_{0}^{2\pi}$$

$$= \frac{1}{2\pi} \left[\frac{-1}{n+1} (2\pi) \right] + \frac{1}{2\pi(n+1)^2} [\sin 2(n+1)\pi - 0]$$

$$= \frac{-1}{n+1} + \frac{1}{2\pi(n+1)^2} [0 - 0] = \underline{\underline{\frac{-1}{n+1}}} \quad \left(\because \cos 2(n+1)\pi = 1, \forall n \in \mathbb{Z} \right)$$

$$\sin 2(n+1)\pi = 0$$

$$\therefore \underline{\underline{I_1 = \frac{-1}{n+1}}}$$

To find I_2 : $I_2 = \frac{1}{2\pi} \int_0^{2\pi} x \sin(n-1)x dx = \frac{1}{2\pi} \left[x \left(\frac{-\cos(n-1)x}{n-1} \right) - \int_0^{2\pi} \left(\frac{-\cos(n-1)x}{n-1} \right) dx \right]$

$$= \frac{1}{2\pi} \left[\frac{-1}{n-1} (2\pi \cos 2(n-1)\pi - 0) \right] + \frac{1}{2\pi(n-1)} \left(\frac{\sin(n-1)x}{n-1} \right)_{0}^{2\pi}$$

$$= \frac{1}{2\pi} \left[\frac{-1}{n-1} (2\pi - 0) \right] + \frac{1}{2\pi(n-1)^2} [\sin 2(n-1)\pi - 0]$$

$$= \frac{-1}{n-1} + \frac{1}{2\pi(n-1)^2} (0 - 0) = \underline{\underline{\frac{-1}{n-1}}}$$

② $\Rightarrow a_n = I_1 - I_2 = \frac{-1}{n+1} - \left(\frac{-1}{n-1} \right) = \frac{1}{n-1} - \frac{1}{n+1} = \frac{n+1-n-1}{n^2-1} = \frac{2}{n^2-1}$

$$\therefore \boxed{a_n = \frac{2}{n^2-1}}, \text{ if } n \neq 1$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_0^{2\pi} x \sin x \sin nx dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x (2 \sin nx \sin x) dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x [\cos(n-1)x - \cos(n+1)x] dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x \cos(n-1)x dx - \frac{1}{2\pi} \int_0^{2\pi} x \cos(n+1)x dx$$

$$b_n = I_1 - I_2 \quad \text{--- ③}$$

To find I_1 : $I_1 = \frac{1}{2\pi} \int_0^{2\pi} x \cos(n-1)x dx = \frac{1}{2\pi} \left[x \frac{\sin(n-1)x}{n-1} - \int_0^{2\pi} \frac{\sin(n-1)x}{n-1} dx \right]$

$$= \frac{1}{2\pi} \left[\frac{1}{n-1} (2\pi \sin 2(n-1)\pi - 0) \right] - \frac{1}{2\pi(n-1)} \left(\frac{-\cos(n-1)x}{n-1} \right)_{0}^{2\pi}$$

$$= \frac{1}{2\pi} \left[\frac{1}{n-1} (2\pi - 0) \right] + \frac{1}{2\pi(n-1)^2} [\cos 2(n-1)\pi - 1]$$

$$= \frac{1}{n-1} + \frac{1}{2\pi(n-1)^2} (0 - 1) = \frac{1}{n-1} - \frac{1}{2\pi(n-1)^2} = \underline{\underline{0}}, \text{ if } n \neq 1.$$

To find $I_2: I_2 = \frac{1}{2\pi} \int_0^{2\pi} x \cos(n+1)x \, dx$

$$= \frac{1}{2\pi} \left[\left(x \frac{\sin(n+1)x}{n+1} \right)_0^{2\pi} - \int_0^{2\pi} \left(\frac{\sin(n+1)x}{n+1} \right) dx \right]$$

$$= \frac{1}{2\pi} \left[\frac{1}{n+1} (2\pi \sin 2(n+1)\pi - 0) \right] - \frac{1}{2\pi(n+1)} \left(\frac{-\cos(n+1)x}{n+1} \right)_0^{2\pi}$$

$$= \frac{1}{2\pi} \left[\frac{1}{n+1} (0-0) \right] + \frac{1}{2\pi(n+1)^2} [\cos 2(n+1)\pi - 1]$$

$$= 0 + \frac{1}{2\pi(n+1)^2} [0-1] = \underline{0}, \text{ if } n \neq 1$$

③ $\Rightarrow b_n = 0 - 0 = 0 \Rightarrow \boxed{b_n = 0}, \text{ if } n \neq 1$

We got $a_n = \frac{2}{n^2-1}, \text{ if } n \neq 1$

If $n=1$, we have $a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx$

$$a_1 = \frac{1}{\pi} \int_0^{2\pi} x \sin x \cos x \, dx = \frac{1}{2\pi} \int_0^{2\pi} x (2 \sin x \cos x) \, dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x \sin 2x \, dx = \frac{1}{2\pi} \left[\left(x \left(-\frac{\cos 2x}{2} \right) \right)_0^{2\pi} - \int_0^{2\pi} \left(-\frac{\cos 2x}{2} \right) dx \right]$$

$$= \frac{1}{2\pi} \left[\frac{-1}{2} (2\pi \cos 4\pi - 0) \right] + \frac{1}{\pi} \left(\frac{\sin 2x}{2} \right)_0^{2\pi}$$

$$= \frac{1}{2\pi} \left[\frac{-1}{2} (2\pi - 0) \right] + \frac{1}{8\pi} [\sin 4\pi - 0]$$

$$= \frac{1}{2\pi} (-\pi) + 0 = \underline{\underline{-\frac{1}{2}}} \Rightarrow \boxed{a_1 = -\frac{1}{2}}$$

④ $\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n$

We got $b_n = 0$, for $n \neq 1$

$\therefore$ If $n=1$, we have $b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx$

$$b_1 = \frac{1}{\pi} \int_0^{2\pi} x \sin x \sin x \, dx = \frac{1}{\pi} \int_0^{2\pi} x \sin^2 x \, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \left(\frac{1 - \cos 2x}{2} \right) dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x \, dx - \frac{1}{2\pi} \int_0^{2\pi} x \cos 2x \, dx$$

$$= \frac{1}{2\pi} \left(\frac{x^2}{2} \right)_0^{2\pi} - \frac{1}{2\pi} \left[x \sin \frac{2x}{2} \right]_0^{2\pi} - \int_0^{2\pi} \frac{\sin 2x}{2} dx$$

$$= \frac{1}{4\pi} (4\pi^2) - \frac{1}{2\pi} \left[\frac{1}{2} (8\pi \sin 4\pi - 0) \right] + \frac{1}{4\pi} \left(-\frac{\cos 2x}{2} \right)_0^{2\pi}$$

$$= \pi - \frac{1}{4\pi} (0 - 0) + \frac{1}{8\pi} [\cos 4\pi - \cos 0] = \pi - \frac{1}{8\pi} (1 - 1) = \pi$$

$$\therefore \boxed{b_1 = \pi}$$

$$\textcircled{1} \Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= -\frac{2}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx + b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

$$= -1 + \frac{1}{2} \cos x + \sum_{n=2}^{\infty} \left(\frac{2}{n^2-1} \right) \cos nx + \pi \sin x + \sum_{n=2}^{\infty} (0) \sin nx$$

$$\therefore f(x \sin x) = -1 + \frac{1}{2} \cos x + \pi \sin x + \frac{2}{3} \cos 2x + \frac{2}{8} \cos 3x + \frac{2}{15} \cos 4x + \dots, \text{ is the required fourier series}$$



Convergence of Fourier Series: If $x=a$ is the point of continuity

of $f(x)$ then the fourier series of $f(x)$ is converged to $f(a)$.

If $x=a$ is a point of discontinuity of $f(x)$ then the

fourier series of $f(x)$ is converged to $\frac{1}{2} [f(a^-) + f(a^+)]$,

where $f(a^-)$ = left hand limit, $f(a^+)$ = Right hand limit.

1. Find the fourier series to represent the function, $f(x)$ is given by $f(x) = x$, $0 \leq x \leq \pi$ & hence deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

Given, $f(x) = x$, $0 \leq x \leq \pi$

$$= 2\pi - x, \pi < x \leq 2\pi$$

Fourier series for a period of 2π is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx, \quad 0 \leq x < \pi$$

$$\text{where } a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx = \frac{1}{\pi} \left[\int_0^{\pi} x dx + \int_{\pi}^{2\pi} (2\pi - x) dx \right]$$

$$= \frac{1}{\pi} \left[\frac{\pi^2}{2} + 2\pi \left(\frac{x^2}{2} \right)_{\pi} - \left(\frac{x^2}{2} \right)_{\pi}^{2\pi} \right] = \frac{1}{\pi} \left[\frac{\pi^2}{2} + 2\pi^2 - \frac{4\pi^2}{2} + \frac{\pi^2}{2} \right]$$

$$= \frac{1}{\pi} [\pi^2 + 2\pi^2 - 2\pi^2] = \pi \quad \Rightarrow \quad \boxed{a_0 = \pi}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \left[\int_0^{\pi} f(x) \cos nx \, dx + \int_{\pi}^{2\pi} f(x) \cos nx \, dx \right]$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} x \cos nx \, dx + \int_{\pi}^{2\pi} (2\pi - x) \cos nx \, dx \right]$$

$$= \frac{1}{\pi} \left[\left(x \frac{\sin nx}{n} \right)_0^{\pi} - \int_0^{\pi} \left(\frac{\sin nx}{n} \right) dx + \left((2\pi - x) \frac{\sin nx}{n} \right)_{\pi}^{2\pi} - \int_{\pi}^{2\pi} \left(-1 \right) \frac{\sin nx}{n} dx \right]$$

$$= \frac{1}{\pi} \left[\frac{1}{n} (\pi \sin n\pi - 0) + \frac{1}{n} \left(\frac{\cos nx}{n} \right)_0^{\pi} + \frac{1}{n} (0 - \pi \sin n\pi) - \frac{1}{n} \left(\frac{\cos nx}{n} \right)_{\pi}^{2\pi} \right]$$

$$= 0 + \frac{1}{n^2\pi} (\cos n\pi - \cos 0) + 0 - \frac{1}{n^2\pi} (\cos 2n\pi - \cos n\pi)$$

$$= \frac{1}{n^2\pi} (\cos n\pi - 1) - \frac{1}{n^2\pi} (1 - \cos n\pi) = \frac{1}{n^2\pi} (\cos n\pi - 1) + \frac{1}{n^2\pi} (\cos n\pi - 1)$$

$$= \frac{2}{n^2\pi} [(-1)^n - 1]$$

$$\begin{aligned} \sin n\pi &= 0 \\ \cos n\pi &= (-1)^n \\ \cos 2n\pi &= (-1)^{2n} = 1 \end{aligned}$$

$$\therefore a_n = 0 \quad \text{for } n \text{ is even}$$

$$= \frac{-4}{n^2\pi} \quad \text{for } n \text{ is odd.}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \left[\int_0^{\pi} x \sin nx \, dx + \int_{\pi}^{2\pi} (2\pi - x) \sin nx \, dx \right]$$

$$= \frac{1}{\pi} \left[\left(x \left(-\frac{\cos nx}{n} \right) \right)_0^{\pi} - \int_0^{\pi} \left(-\frac{\cos nx}{n} \right) dx + \left((2\pi - x) \left(-\frac{\cos nx}{n} \right) \right)_{\pi}^{2\pi} - \int_{\pi}^{2\pi} \left(-1 \right) \left(-\frac{\cos nx}{n} \right) dx \right]$$

$$= \frac{1}{\pi} \left[-\frac{1}{n} [\pi \cos n\pi - 0] + \frac{1}{n} \left(\frac{\sin nx}{n} \right)_0^{\pi} + \frac{1}{n} \left(-\frac{1}{n} \right) (0 - \pi \cos n\pi) - \frac{1}{n} \left(\frac{\sin nx}{n} \right)_{\pi}^{2\pi} \right]$$

$$= -\frac{1}{n} \cos n\pi + \frac{1}{n^2\pi} (\sin n\pi - 0) + \frac{1}{n} \cos n\pi - \frac{1}{n^2\pi} (\sin 2n\pi - \sin n\pi)$$

$$= \frac{1}{n^2\pi} (0 - 0) - \frac{1}{n^2\pi} (0 - 0) = 0 \quad \Rightarrow \quad \boxed{b_n = 0}$$

$$\textcircled{1} \Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} 0 \sin nx$$

$$= \frac{\pi}{2} + \sum_{n=1,3,5,\dots}^{\infty} \frac{-4}{n^2 \pi} \cos nx + 0 = \frac{\pi}{2} - \frac{4}{\pi} \left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right) \quad \text{--- (2)}$$

Deduction: At $x=0$, $f(x)$ is continuous.

Hence the fourier series is converged to $f(0)$.

put $x=0$ in (2), we get $f(0) = \frac{\pi}{2} - \frac{4}{\pi} \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right)$

$$\Rightarrow 0 = \frac{\pi}{2} - \frac{4}{\pi} \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right)$$

$$\Rightarrow \frac{\pi}{2} = \frac{4}{\pi} \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right) \Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

Ex 15

Q. Find the fourier series of $f(x) = x + x^2$, $-\pi < x < \pi$ & hence

deduce the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}$.

sol. given $f(x) = x + x^2$, $-\pi < x < \pi$

The fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x dx + \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx$$

$$= 0 + \frac{2}{\pi} \int_0^{\pi} x^2 dx$$

$$= \frac{2}{3\pi} (x^3)_0^{\pi} = \frac{2\pi^3}{3\pi} = \frac{2\pi^2}{3}$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

$$\left(\because \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx, f(x) \text{ is even} \right)$$

$$= 0, f(x) \text{ is odd}$$

$$f(x) = -f(x) \rightarrow \text{odd}$$

$$f(x) = f(x) \rightarrow \text{even}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) \cos nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos nx dx + \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx dx$$

$$= 0 + \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx$$

($\because x \cos nx$ is odd function

$x^2 \cos nx$ is even function)

$$= \frac{2}{\pi} \left[\left(\frac{x^2 \sin nx}{n} \right)_0^{\pi} - \int_0^{\pi} (2x) \frac{\sin nx}{n} dx \right]$$

$$= \frac{2}{\pi} \left[\frac{1}{n} (\pi^2 \sin n\pi - 0) \right] - \frac{4}{n\pi} \int_0^{\pi} x \sin nx dx$$

$$= 0 - \frac{4}{n\pi} \left[x \left(-\frac{\cos nx}{n} \right) \right]_0^\pi - \int_0^\pi \left(-\frac{\cos nx}{n} \right) dx$$

$$= \frac{-4}{n\pi} \left[\frac{-1}{n} (\pi \cos n\pi - 0) \right] - \frac{4}{n^2 \pi} \left(\frac{\sin n\pi}{n} \right)_0^\pi$$

$$\frac{+4}{n^2} (-1)^n - \frac{4}{n^3 \pi} (\sin n\pi - 0) = \frac{+4}{n^2} (-1)^n - 0$$

$$a_n = \frac{4}{5^2} (-1)^n$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x+x^2) \sin nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx \, dx + \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx + 0$$

$$= \frac{1}{\pi} \left[\left(x \left(-\frac{\cos nx}{n} \right) \right)_0^{\pi} - \int_0^{\pi} \left(-\frac{\cos nx}{n} \right) dx \right]$$

$$= \frac{2}{\pi} \left[-\frac{1}{n} (\pi \cos n\pi - 0) \right] + \frac{2}{n\pi} \left(\frac{\sin n\pi}{n} \right)_0^\pi$$

$$= -\frac{2}{n} (-1)^n + \frac{2}{n+\pi} (\sin n\pi - 0) = -\frac{2}{n} (-1)^n + 0$$

$$= (-1)^2 \frac{2(-1)^n}{2} = \frac{1}{2} 2 (-1)^{n+1}$$

$$b_n = \frac{2}{n} (-1)^{n+1}$$

$$\textcircled{1} \Rightarrow f(x) = \left(\frac{2\pi^2}{3}\right) + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos nx + \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin nx$$

$$x+x^2 = \frac{\pi^2}{3} + 4 \left[\frac{-\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \dots \right] + 2 \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \right] - 0$$

Deduction: At $x=0$, $f(x)$ is continuous. Hence the fourier series converges to $f(0)$.

put $x=0$ in ②, $0 = \frac{\pi^2}{3} + 4 \left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \dots \right]$

$$\frac{\pi^2}{3} = 4 \left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots \right]$$

$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}$$

3. Find a Fourier Series to represent a function, $f(x) = e^x$ in

$$-\pi < x < \pi \quad \& \quad \text{hence deduce the series } 2 \left[\frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} - \dots \right] = \frac{\pi}{\sinh \pi}$$

Given $f(x) = e^x, -\pi < x < \pi$

The Fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x dx = \frac{1}{\pi} (e^{\pi} - e^{-\pi}) = \frac{2}{\pi} \left(\frac{e^{\pi} - e^{-\pi}}{2} \right) = \frac{2}{\pi} \sinh \pi$$

$$\boxed{a_0 = \frac{2}{\pi} \sinh \pi}$$

$$\left(\because \sinh x = \frac{e^x - e^{-x}}{2} \right)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \cos nx dx$$

WKT $\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx)$

$$= \frac{1}{\pi} \left[\frac{e^x}{n^2+1} (\cos nx + n \sin nx) \right]_{-\pi}^{\pi} \quad \begin{matrix} (a=1) \\ b=n \end{matrix}$$

$$= \frac{1}{\pi(n^2+1)} \left[e^{\pi} (\cos n\pi + n \sin n\pi) - e^{-\pi} (\cos n\pi - n \sin n\pi) \right]$$

$$= \frac{1}{\pi(n^2+1)} \left[e^{\pi} \cos n\pi - e^{-\pi} \cos n\pi \right] = \frac{\cos n\pi}{\pi(n^2+1)} (e^{\pi} - e^{-\pi})$$

$$= \frac{2 \cos n\pi}{\pi(n^2+1)} \left(\frac{e^{\pi} - e^{-\pi}}{2} \right) = \frac{2(-1)^n \sinh \pi}{\pi(n^2+1)}$$

$$\boxed{a_n = \frac{2 \sinh \pi}{\pi(n^2+1)} (-1)^n}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \sin nx dx$$

WKT $\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx)$

$$= \frac{1}{\pi} \left[\frac{e^x}{n^2+1} (\sin nx - n \cos nx) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(n^2+1)} \left[e^{\pi} (\sin n\pi - n \cos n\pi) - e^{-\pi} (-\sin n\pi - n \cos n\pi) \right]$$

$$= \frac{1}{\pi(n^2+1)} \left[e^{\pi} \sin n\pi - n e^{\pi} \cos n\pi + n e^{-\pi} \cos n\pi \right]$$

$$= \frac{-n \cos n\pi}{\pi(n^2+1)} (e^{\pi} - e^{-\pi}) = \frac{-2n(-1)^n}{\pi(n^2+1)} \left(\frac{e^{\pi} - e^{-\pi}}{2} \right)$$

$$(\because \cos n\pi = (-1)^n)$$

$$= \frac{-2n(-1)^n \sinh \pi}{\pi(n^2+1)} = \frac{(-1)^{n+1} 2 \sinh \pi}{\pi(n^2+1)}$$

$$\therefore \boxed{b_n = \frac{+2n \sinh \pi}{\pi(n^2+1)} (-1)^{n+1}}$$

$$\textcircled{1} \Rightarrow f(x) = \frac{\frac{2}{\pi} \sinh \pi}{2} + \sum_{n=1}^{\infty} \frac{2 \sinh \pi}{\pi(n^2+1)} (-1)^n \cos nx + \sum_{n=1}^{\infty} \frac{+2n \sinh \pi}{\pi(n^2+1)} (-1)^{n+1} \sin nx$$

$$e^x = \frac{1}{\pi} \sinh \pi + \frac{2 \sinh \pi}{\pi} \left[-\frac{\cos x}{1^2+1} + \frac{\cos 2x}{2^2+1} - \frac{\cos 3x}{3^2+1} + \dots \right]$$

$$+ \frac{2 \sinh \pi}{\pi} \left[\frac{\sin x}{1^2+1} - \frac{2 \sin 2x}{2^2+1} + \frac{3 \sin 3x}{3^2+1} - \dots \right]$$

$$e^x = \frac{\sinh \pi}{\pi} \left[1 + 2 \left(-\frac{\cos x}{2} + \frac{\cos 2x}{2^2+1} - \frac{\cos 3x}{3^2+1} + \frac{\cos 4x}{4^2+1} - \dots \right) \right]$$

$$+ \frac{2 \sinh \pi}{\pi} \left[\frac{\sin x}{\pi} - \frac{2 \sin 2x}{2^2+1} + \frac{3 \sin 3x}{3^2+1} - \dots \right]$$

Deduction: At $x=0$, $f(x)$ is continuous. Hence the Fourier

series converges to $f(0)$

put $x=0$ in $\textcircled{2}$, we get

$$e^0 = \frac{\sinh \pi}{\pi} \left[1 + 2 \left(-\frac{1}{2} + \frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} - \dots \right) \right] + 0$$

$$1 = \frac{\sinh \pi}{\pi} \left[1 - 1 + 2 \left(\frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} - \dots \right) \right]$$

$$\Rightarrow 2 \left[\frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} - \dots \right] = \frac{\pi}{\sinh \pi}$$

sol/s

4. Find the Fourier series of $f(x) = 0, -\pi < x < 0$ & hence

deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

Qd. given $f(x) = 0, -\pi < x < 0$
 $= \frac{\pi x}{4}, 0 \leq x < \pi$

The fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

where $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \left[\int_{-\pi}^0 f(x) dx + \int_0^{\pi} f(x) dx \right]$

$$= \frac{1}{\pi} \int_{-\pi}^0 0 dx + \frac{1}{\pi} \int_0^{\pi} \left(\frac{\pi x}{4} \right) dx = \frac{1}{4} \left(\frac{x^2}{2} \right)_0^{\pi} = \frac{1}{8} (\pi^2) = \frac{\pi^2}{8}$$

$$\therefore \boxed{a_0 = \frac{\pi^2}{8}}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \left[\int_{-\pi}^0 0 \cos nx dx + \int_0^{\pi} \frac{\pi x}{4} \cos nx dx \right]$$

$$= \frac{1}{4} \int_0^{\pi} x \cos nx dx = \frac{1}{4} \left[\left(x \frac{\sin nx}{n} \right)_0^{\pi} - \int_0^{\pi} \frac{\sin nx}{n} dx \right]$$

$$= \frac{1}{4n} (\sin n\pi - 0) - \frac{1}{4n} \left(\frac{\cos nx}{n} \right)_0^{\pi} = 0 + \frac{1}{4n^2} (\cos n\pi - 1)$$

$$= \frac{1}{4n^2} ((-1)^n - 1)$$

$\therefore a_n = 0$, for n is even

$a_n = \frac{-1}{2n^2}$, for n is odd

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \left[\int_{-\pi}^0 0 \sin nx dx + \int_0^{\pi} \frac{\pi x}{4} \sin nx dx \right]$$

$$= \frac{1}{4} \int_0^{\pi} x \sin nx dx = \frac{1}{4} \left[\left(x \left(-\frac{\cos nx}{n} \right) \right)_0^{\pi} - \int_0^{\pi} \left(-\frac{\cos nx}{n} \right) dx \right]$$

$$= -\frac{1}{4n} (\pi \cos n\pi - 0) + \frac{1}{4n} \left(\frac{\sin nx}{n} \right)_0^{\pi} = -\frac{1}{4n} (\pi (-1)^n) + \frac{1}{4n^2} (\sin n\pi - 0)$$

$$= -\frac{\pi}{4n} (-1)^n = \frac{\pi}{4n} (-1)^{n+1}$$

$$\therefore \boxed{b_n = \frac{\pi}{4n} (-1)^{n+1}}$$

$$\textcircled{1} \Rightarrow f(x) = \left(\frac{\pi^2}{8} \right) + \sum_{n=1,3,5,\dots}^{\infty} \left(\frac{-1}{2n^2} \right) \cos nx + \sum_{n=1}^{\infty} \frac{\pi}{4n} (-1)^{n+1} \sin nx$$

$$= \frac{\pi^2}{16} - \frac{1}{2} \left[\frac{\cos x}{1^2} + \frac{\cos 2x}{3^2} + \frac{\cos 3x}{5^2} + \dots \right] + \frac{\pi}{4} \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \right] \quad \text{--- (2)}$$

Deduction: At $x=0$, $f(x)$ is continuous.

Hence the Fourier series of $f(x)$ converges to $f(0)$.

Put $x=0$ in (2), we get

$$f(0) = \frac{\pi^2}{16} - \frac{1}{2} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right] \Rightarrow 0 = \frac{\pi^2}{16} - \frac{1}{2} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

$$\Rightarrow \frac{\pi^2}{16} = \frac{1}{2} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

$$\Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

5) If $f(x) = x$ in $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $f(x) = 0$ in $(\frac{\pi}{2}, \frac{3\pi}{2})$ find the Fourier series of $f(x)$. Deduce that $\frac{\pi^r}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^r}$

Sol:- The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

$$a_0 = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x) dx = \frac{1}{\pi} \left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 0 dx \right]$$

$$= \frac{1}{\pi} \left[\frac{x^2}{2} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{2\pi} \left[\frac{\pi^2}{4} - \frac{\pi^2}{4} \right] = 0.$$

$$a_n = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x) \cos nx dx = \frac{1}{\pi} \left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \cos nx dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 0 \cos nx dx \right]$$

$$= \frac{1}{\pi} \left[x \frac{\sin nx}{n} - (1) \left(\frac{-\cos nx}{n^2} \right) \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \quad \left[\because \int u v dx = u v_1 - u' v_2 + \dots \right]$$

$$= \frac{1}{\pi} \left[\frac{\pi/2 \sin n\pi/2}{n} + \frac{\cos n\pi/2}{n^2} - \left(\frac{-\pi/2 \sin n(-\pi/2)}{n} + \frac{\cos n(-\pi/2)}{n^2} \right) \right]$$

$$a_n = 0$$

$$\& b_n = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x) \sin nx dx = \frac{1}{\pi} \left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \sin nx dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 0 (\sin nx) dx \right]$$

$$= \frac{1}{\pi} \left[x \left(\frac{-\cos nx}{n} \right) - (1) \left(\frac{-\sin nx}{n^2} \right) \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{\pi} \left[-\frac{\pi/2 \cos n\pi/2}{n} + \frac{\sin n\pi/2}{n^2} - \left(\frac{-\pi/2 \cos n(-\pi/2)}{n} - \frac{\sin n(-\pi/2)}{n^2} \right) \right]$$

$$= \frac{1}{\pi} \left[-\frac{\pi}{n} \cos n\pi/2 + \frac{2}{n^2} \sin n\pi/2 \right]$$

$$\boxed{b_n = -\frac{1}{n} \cos \frac{n\pi}{2} + \frac{2}{n^2 \pi} \sin n\pi/2.}$$

$$\therefore f(x) = 0 + 0 + \sum_{n=1}^{\infty} \left(-\frac{\cos n\pi/2}{n} + \frac{2}{n^2 \pi} \sin n\pi/2 \right) \sin nx.$$

$$\text{i.e. } f(x) = \left[-\frac{\cos \pi/2}{1} + \frac{2}{1^2 \pi} \sin \pi/2 \right] \sin x + \left[-\frac{\cos \pi}{2} + \frac{2}{2^2 \pi} \sin \pi \right] \sin 2x$$

$$+ \left[-\frac{\cos 3\pi/2}{3} + \frac{2}{3^2 \pi} \sin 3\pi/2 \right] \sin 3x + \dots$$

$$= \left[-0 + \frac{2}{\pi} (1) \right] \sin x + \left[-(-\frac{1}{2}) + 0 \right] \sin 2x +$$

$$\left[0 + \frac{2}{3^2 \pi} (-1) \right] \sin 3x + \dots$$

$$\therefore f(x) = \frac{2}{\pi} \frac{1}{1^r} \sin x + \frac{1}{2} \sin 2x - \frac{2}{\pi 3^r} \sin 3x + \dots \quad (2)$$

At $x = \frac{\pi}{2}$ which is a point of discontinuity $\left[\begin{array}{l} \lim_{x \rightarrow \frac{\pi}{2}^-} f(x) = \frac{\pi}{2} \\ \lim_{x \rightarrow \frac{\pi}{2}^+} f(x) = 0 \end{array} \right]$

$$f\left(\frac{\pi}{2}\right) = \frac{1}{2} \left[f\left(\frac{\pi}{2}^-\right) + f\left(\frac{\pi}{2}^+\right) \right]$$

$$= \frac{1}{2} \left[\frac{\pi}{2} + 0 \right] = \frac{\pi}{4}$$

$$\left[L.H.L \neq R.H.L \right]$$

Putting $x = \frac{\pi}{2}$ in (2), we get

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{4} = \frac{2}{\pi} \frac{1}{1^r} \sin \frac{\pi}{2} + \frac{1}{2} \sin \pi - \frac{2}{\pi 3^r} \sin \frac{3\pi}{2} + \dots$$

$$\text{i.e. } \frac{\pi}{4} = \frac{2}{\pi} \frac{1}{1^r} (1) + 0 - \frac{2}{\pi 3^r} (-1) + \dots$$

$$\Rightarrow \frac{\pi}{4} = \frac{2}{\pi} \left[\frac{1}{1^r} + \frac{1}{3^r} + \dots \right]$$

$$\Rightarrow \frac{\pi^2}{8} = \frac{1}{1^r} + \frac{1}{3^r} + \dots$$

$$\therefore \frac{1}{1^r} + \frac{1}{3^r} + \dots = \frac{\pi^2}{8}$$

$$\text{i.e. } \sum_{n=1}^{\infty} \frac{1}{(2n-1)^r} = \frac{\pi^2}{8}$$

6) Find the F.S. for $f(x) = -\pi$, for $-\pi < x < 0$
 $= x$, for $0 < x < \pi$ &

hence deduce that $\frac{1}{1^r} + \frac{1}{3^r} + \frac{1}{5^r} + \dots = \frac{\pi^2}{8}$

7) Obtain the F.S. expansion of $f(x) = \frac{\pi-x}{2a}$ in $(0, 2\pi)$ &

Deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

8) Find the F.S. of $f(x) = 0$, when $-\pi \leq x \leq 0$
 $= x^r$, when $0 \leq x \leq \pi$

Change of interval:

Fourier Series for all periodic functions, where l is a +ve constant ~~arbitrary~~:

Let the function $f(x)$ be periodic of period ' $2l$ '. The Fourier series of $f(x)$ in the interval $[c, c+2l]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right), \text{ where}$$

$$a_0 = \frac{1}{l} \int_c^{c+2l} f(x) dx, \quad a_n = \frac{1}{l} \int_c^{c+2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx,$$

$$b_n = \frac{1}{l} \int_c^{c+2l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

Q1/5/15

1 Find the Fourier series of periodicity 3 for $f(x) = 2x - x^2$ in $0 < x < 3$.

sol) Given $f(x) = 2x - x^2$, $0 < x < 3$

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\text{Here } c = 0, \quad c+2l = 3 \Rightarrow 2l = 3 \Rightarrow \boxed{l = \frac{3}{2}}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi}{3}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi}{3}x\right) \quad \text{--- ①}$$

$$a_0 = \frac{1}{l} \int_c^{c+2l} f(x) dx$$

$$= \frac{2}{3} \int_0^3 (2x - x^2) dx = \frac{2}{3} \left[2 \left(\frac{x^2}{2} \right)_0^3 - \left(\frac{x^3}{3} \right)_0^3 \right] = \frac{2}{3} \left[9 - \frac{27}{3} \right]$$

$$= \frac{2}{3} (9 - 9) = 0 \Rightarrow \boxed{a_0 = 0}$$

$$a_n = \frac{1}{l} \int_c^{c+l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{3} \int_0^3 (2x - x^2) \cos\left(\frac{2n\pi}{3} x\right) dx$$

$$= \frac{2}{3} \left[\left((2x - x^2) \frac{\sin\left(\frac{2n\pi}{3} x\right)}{\left(\frac{2n\pi}{3}\right)} \right)_0^3 - \int_0^3 (2 - 2x) \frac{\sin\left(\frac{2n\pi}{3} x\right)}{\frac{2n\pi}{3}} dx \right]$$

$$= \frac{2}{3} \times \frac{3}{2n\pi} [(-3) \sin 2n\pi - 0] - \frac{2}{3} \times \frac{3}{2n\pi} \times \frac{3}{2} \int_0^3 (1-x) \sin\left(\frac{2n\pi}{3} x\right) dx$$

$$= 0 - \frac{2}{n\pi} \left[\left((1-x) \left(\frac{-\cos\left(\frac{2n\pi}{3} x\right)}{\left(\frac{2n\pi}{3}\right)} \right) \right)_0^3 - \int_0^3 (-1) \frac{-\cos\left(\frac{2n\pi}{3} x\right)}{\left(\frac{2n\pi}{3}\right)} dx \right]$$

$$= -\frac{2}{n\pi} \times \frac{-3}{2n\pi} [(-2) \cos 2n\pi - 1 \cdot \cos 0] + \frac{2}{n\pi} \times \frac{3}{2n\pi} \left(\frac{\sin\left(\frac{2n\pi}{3} x\right)}{\left(\frac{2n\pi}{3}\right)} \right)_0^3$$

$$= \frac{+3}{n^2 \pi^2} [-2(+1) - 1] + \frac{3}{n^2 \pi^2} \times \frac{3}{2n\pi} [\sin 2n\pi - 0]$$

$$= \frac{-9}{n^2 \pi^2} + 0 \Rightarrow \boxed{a_n = \frac{-9}{n^2 \pi^2}}$$

$$b_n = \frac{1}{l} \int_c^{c+l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{3} \int_0^3 (2x - x^2) \sin\left(\frac{2n\pi}{3} x\right) dx$$

$$= \frac{2}{3} \left[\left((2x - x^2) \left(\frac{-\cos\left(\frac{2n\pi}{3} x\right)}{\left(\frac{2n\pi}{3}\right)} \right) \right)_0^3 - \int_0^3 (2 - 2x) \left(\frac{-\cos\left(\frac{2n\pi}{3} x\right)}{\frac{2n\pi}{3}} \right) dx \right]$$

$$= \frac{2}{3} \times \frac{-3}{2n\pi} [(-3) \cos 2n\pi - 0] + \frac{2}{3} \times \frac{3}{2n\pi} \times \frac{2}{3} \int_0^3 (1-x) \cos\left(\frac{2n\pi}{3} x\right) dx$$

$$= \frac{-1}{n\pi} [(-3) \cdot 1] + \frac{2}{n\pi} \left[\left((1-x) \frac{\sin\left(\frac{2n\pi}{3} x\right)}{\left(\frac{2n\pi}{3}\right)} \right)_0^3 - \int_0^3 (-1) \frac{\sin\left(\frac{2n\pi}{3} x\right)}{\frac{2n\pi}{3}} dx \right]$$

$$= \frac{3}{n\pi} + \frac{2}{n\pi} \times \frac{3}{2n\pi} [(-2) \sin 2n\pi - 0] + \frac{2}{n\pi} \times \frac{3}{2n\pi} \left(\frac{-\cos\left(\frac{2n\pi}{3} x\right)}{\frac{2n\pi}{3}} \right)_0^3$$

$$= \frac{3}{n\pi} + \frac{3}{n^2 \pi^2} (0) - \frac{3}{n^2 \pi^2} \times \frac{3}{2n\pi} [\cos 2n\pi - 1]$$

$$= \frac{3}{n\pi} - \frac{9}{2n^3\pi^3} (x-\pi) = \frac{3}{n\pi} - 0 = \frac{3}{n\pi} \Rightarrow \boxed{b_n = \frac{3}{n\pi}}$$

$$\textcircled{1} \Rightarrow f(x) = \frac{0}{2} + \sum_{n=1}^{\infty} \frac{-9}{n^3\pi^3} \cos\left(\frac{2n\pi}{3}\right)x + \sum_{n=1}^{\infty} \frac{3}{n\pi} \sin\left(\frac{2n\pi}{3}\right)x$$

Q2/s15

Q. Find the fourier series for $f(x) = 2lx - x^2$ in $(0, 2l)$, hence show that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{12}$

Sol. given $f(x) = 2lx - x^2$ in $(0, 2l)$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ — (1) be the fourier series

$$a_0 = \frac{1}{l} \int_0^{2l} f(x) dx = \frac{1}{l} \int_0^{2l} (2lx - x^2) dx = \frac{1}{l} \left[\frac{2l}{2} (x^2)_0^{2l} - \frac{1}{3} (x^3)_0^{2l} \right]$$

$$= \frac{1}{l} \left[l(4l^2) - \frac{1}{3}(8l^3) \right] = \frac{1}{l} \left[4l^3 - \frac{8l^3}{3} \right] = \frac{1}{l} \times \frac{4l^3}{3} = \frac{4l^2}{3}$$

$$\therefore \boxed{a_0 = \frac{4l^2}{3}}$$

$$a_n = \frac{1}{l} \int_0^{2l} f(x) \cos \frac{n\pi x}{l} dx = \frac{1}{l} \int_0^{2l} (2lx - x^2) \cos \frac{n\pi x}{l} dx$$

$$= \frac{1}{l} \left[\left((2lx - x^2) \frac{\sin \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)} \right)_0^{2l} - \int_0^{2l} (2l - 2x) \frac{\sin \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)} dx \right]$$

$$= \frac{1}{l} \left[\frac{l}{n\pi} (0 - 0) - \frac{1}{l} \times \frac{l}{n\pi} \times 2 \int_0^{2l} (l - x) \sin\left(\frac{n\pi x}{l}\right) dx \right]$$

$$= \left[-\frac{2}{n\pi} \left[(l-x) \left(-\frac{\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}} \right) \right]_0^{2l} - \frac{2}{n\pi} \int_0^{2l} (-1) \left(-\frac{\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}} \right) dx \right]$$

$$= -\frac{2}{n\pi} \times \frac{l}{n\pi} \left[(-1) \cos 2n\pi - (-1) \cos 0 \right] + \frac{2}{n\pi} \times \frac{1}{n\pi} \left[\sin\left(\frac{n\pi x}{l}\right) \right]_0^{2l}$$

$$= \frac{2l}{n^2\pi^2} [-1(1) - 1] = -\frac{4l^2}{n^2\pi^2} + \frac{2l^2}{n^3\pi^3} [\sin 2n\pi - 0]$$

$$= -\frac{4l^2}{n^2\pi^2} \quad \left[\begin{array}{l} \sin 2n\pi = 0 \\ \cos 2n\pi = 1 \end{array} \right]$$

$$\therefore \boxed{a_n = -\frac{4l^2}{n^2\pi^2}}$$

$$b_n = \frac{1}{l} \int_0^{2l} f(x) \sin \frac{n\pi x}{l} dx = \frac{1}{l} \int_0^{2l} (2lx - x^2) \sin \frac{n\pi x}{l} dx$$

$$= \frac{1}{l} \left[\left((2lx - x^2) \left(-\frac{\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) \right)_0^{2l} - \int_0^{2l} (2l - 2x) \left(-\frac{\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) dx \right]$$

$$= \frac{1}{l} \left[\frac{-l}{n\pi} (0 - 0) \right] + \frac{2}{n\pi} \int_0^{2l} (l - x) \cos \frac{n\pi x}{l} dx$$

$$= \frac{2}{n\pi} \left[\left((l - x) \frac{\sin \frac{n\pi x}{l}}{\left(\frac{n\pi}{l} \right)} \right)_0^{2l} - \int_0^{2l} (-1) \frac{\sin \frac{n\pi x}{l}}{\left(\frac{n\pi}{l} \right)} dx \right] \left[\begin{array}{l} \because \sin 2n\pi = 0 \\ \cos 2n\pi = 1 \end{array} \right]$$

$$= \frac{2}{n\pi} \left[\frac{l}{n\pi} (-l \sin 2n\pi - l \sin 0) \right] + \frac{2l}{n^2\pi^2} \left(-\cos \frac{n\pi x}{l} \right)_0^{2l}$$

$$= \frac{2l}{n^2\pi^2} (0 - 0) - \frac{2l^2}{n^3\pi^3} (\cos 2n\pi - \cos 0) = 0 - 0 = 0$$

$$\Rightarrow \boxed{b_n = 0}$$

$$\textcircled{1} \Rightarrow f(x) = \frac{2l^2}{3} + \sum_{n=1}^{\infty} -\frac{4l^2}{n^2\pi^2} \cos \frac{n\pi x}{l} + 0$$

$$2lx - x^2 = \frac{8l^2}{3} - \frac{4l^2}{\pi^2} \left[\frac{\cos \frac{\pi x}{l}}{1^2} + \frac{\cos \frac{2\pi x}{l}}{2^2} + \frac{\cos \frac{3\pi x}{l}}{3^2} + \dots \right] \quad \textcircled{2}$$

Deduction: At $x=l$, $f(x)$ is continuous & hence the fourier series of $f(x)$ converges to $f(l)$

put $x=l$ in $\textcircled{2}$, we get

$$2l^2 - l^2 = \frac{2l^2}{3} - \frac{4l^2}{\pi^2} \left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \dots \right]$$

$$l^2 - \frac{2l^2}{3} = \frac{4l^2}{\pi^2} \left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots \right] \Rightarrow \frac{l^2}{3} = \frac{4l^2}{\pi^2} \left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots \right]$$

$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}$$

3. Expand $f(x) = \pi x$, $0 < x < 1$
 $= 0$, $1 < x < 2$ into fourier series.

$$\text{Sol. given } f(x) = \pi x, \quad 0 < x < 1$$

$$= 0, \quad 1 < x < 2$$

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$\text{Here } c=0, \quad c+2l=2 \Rightarrow 0+2l=2 \Rightarrow \boxed{l=1}$$

$$a_0 = \frac{1}{l} \int_c^{c+2l} f(x) dx = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x \quad \textcircled{1}$$

$$a_0 = \frac{1}{l} \int_0^l f(x) dx = \frac{1}{1} \int_0^2 f(x) dx = \int_0^1 \pi x dx + \int_1^2 0 dx$$

$$\Rightarrow \boxed{a_0 = \frac{\pi}{2}}$$

$$a_n = \frac{1}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx = \int_0^2 f(x) \cos n\pi x dx \quad (\because l=1)$$

$$= \int_0^1 \pi x \cos n\pi x dx + \int_1^2 0 \cos n\pi x dx$$

$$= \pi \int_0^1 x \cos n\pi x dx + 0 = \pi \left[\left(x \frac{\sin n\pi x}{n\pi} \right)_0^1 - \int_0^1 \frac{\sin n\pi x}{n\pi} dx \right]$$

$$= \pi \left[\frac{1}{n\pi} (\sin n\pi - 0) \right] - \frac{1}{n} \left(-\frac{\cos n\pi x}{n\pi} \right)_0^1 \quad [\because \sin n\pi = 0]$$

$$a_n = 0 + \frac{1}{n^2\pi} (\cos n\pi - \cos 0) = \frac{1}{n^2\pi} ((-1)^n - 1) \quad [\cos n\pi = (-1)^n]$$

$$a_n = 0, \text{ if } n \text{ is even}$$

$$= -\frac{2}{n^2\pi}, \text{ if } n \text{ is odd}$$

$$b_n = \frac{1}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx = \int_0^2 f(x) \sin n\pi x dx \quad (\because l=1)$$

$$= \int_0^1 \pi x \sin n\pi x dx + \int_1^2 0 \sin n\pi x dx = \pi \int_0^1 x \sin n\pi x dx$$

$$= \pi \left[\left(x \left(-\frac{\cos n\pi x}{n\pi} \right) \right)_0^1 - \int_0^1 \left(-\frac{\cos n\pi x}{n\pi} \right) dx \right]$$

$$= \pi \left[\left(-\frac{1}{n\pi} \right) (\cos n\pi - 0) \right] + \frac{1}{n} \left(\frac{\sin n\pi x}{n\pi} \right)_0^1$$

$$= -\frac{1}{n} \cos n\pi + 0 + \frac{1}{n^2\pi} (\sin n\pi - 0) \Rightarrow$$

$$= \frac{1}{n^2\pi} (-1)^{n+1} + 0 = \frac{1}{n^2\pi} (-1)^{n+1} \Rightarrow \boxed{b_n = \frac{1}{n^2\pi} (-1)^{n+1}}$$

$$\Rightarrow f(x) = \left(\frac{\pi}{2} \right) + \sum_{n=1,3,5,\dots}^{\infty} \frac{-2}{n^2\pi} \cos n\pi x + \sum_{n=1}^{\infty} \frac{1}{n^2\pi} (-1)^{n+1} \sin n\pi x$$

$$= \frac{\pi}{4} - \frac{2}{\pi} \left[\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \dots \right]$$

$$+ \pi^2 \left[\frac{\sin \pi x}{1} - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - \dots \right]$$

23/5/15

4. Find the fourier series expansion for $f(x)=2$, if $-2 \leq x \leq 0$ $=x$, if $0 < x < 2$ b). given $f(x)=2$, if $-2 \leq x \leq 0$ $=x$, if $0 < x < 2$

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \text{--- (1)}$$

$$\text{Here } c = -2, \quad c+2l = 2 \Rightarrow -2+2l=2 \Rightarrow 2l=4 \Rightarrow \boxed{l=2}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2} \quad \text{--- (2)}$$

$$a_0 = \frac{1}{l} \int_{-2}^2 f(x) dx = \frac{1}{2} \left[\int_{-2}^0 2 dx + \int_0^2 x dx \right] = \frac{1}{2} \left[2(x)_0^{-2} + \frac{1}{2}(x^2)_0^2 \right]$$

$$= \frac{1}{2} \left[2(2) + \frac{1}{2}(4) \right] = \frac{6}{2} = 3 \Rightarrow \boxed{a_0=3}$$

$$a_n = \frac{1}{l} \int_{-2}^2 f(x) dx \cos \frac{n\pi x}{l} dx = \frac{1}{2} \left[\int_{-2}^0 f(x) \cos \frac{n\pi x}{2} dx + \int_0^2 x \cos \frac{n\pi x}{2} dx \right]$$

$$= \frac{1}{2} \left[\int_{-2}^0 2 \cos \frac{n\pi x}{2} dx + \int_0^2 x \cos \frac{n\pi x}{2} dx \right]$$

$$= \frac{1}{2} \left[2 \left(\frac{\sin \left(\frac{n\pi x}{2} \right)}{\left(\frac{n\pi}{2} \right)} \right)_{-2}^0 + \left(x \frac{\sin \frac{n\pi x}{2}}{\left(\frac{n\pi}{2} \right)} - \int_0^2 \frac{\sin \frac{n\pi x}{2}}{\left(\frac{n\pi}{2} \right)} dx \right) \right]$$

$$= \frac{2}{n\pi} (0 + \sin n\pi) + \frac{1}{n\pi} (2 \sin n\pi - 0) - \frac{1}{n\pi} \left(\frac{-\cos \left(\frac{n\pi x}{2} \right)}{\left(\frac{n\pi}{2} \right)} \right)_0^2$$

$$= 0 + 0 + \frac{2}{n^2\pi^2} (\cos n\pi - \cos 0) = \frac{2}{n^2\pi^2} ((-1)^n - 1)$$

$$\therefore a_n = 0, \text{ if } n \text{ is even}$$

$$= \frac{-4}{n^2\pi^2}, \text{ if } n \text{ is odd}$$

$$b_n = \frac{1}{l} \int_{-2}^2 f(x) \sin \frac{n\pi x}{l} dx = \frac{1}{2} \int_{-2}^2 f(x) \sin \frac{n\pi x}{2} dx$$

$$= \frac{1}{2} \left[\int_{-2}^0 2 \sin \frac{n\pi x}{2} dx + \int_0^2 x \sin \frac{n\pi x}{2} dx \right]$$

$$= 2 \left(\frac{-\cos \frac{n\pi x}{2}}{\left(\frac{n\pi}{2} \right)} \right)_{-2}^0 + \frac{1}{2} \left[\left(x \left(\frac{-\cos \frac{n\pi x}{2}}{\left(\frac{n\pi}{2} \right)} \right) \right)_0^2 - \int_0^2 \left(\frac{-\cos \frac{n\pi x}{2}}{\left(\frac{n\pi}{2} \right)} \right) dx \right]$$

$$= -\frac{2}{n\pi} (\cos 0 - \cos n\pi) - \frac{1}{n\pi} (2 \cos n\pi - 0) + \frac{1}{n\pi} \left(\frac{\sin(\frac{n\pi x}{2})}{(\frac{n\pi}{2})} \right)^2$$

$$= -\frac{2}{n\pi} (1 - (-1)^n) - \frac{1}{n\pi} (2(-1)^n) + \frac{2}{n^2\pi^2} (\sin n\pi - 0)$$

$$= \frac{-2 + 2(-1)^n - 2(-1)^n + 0}{n\pi} = -\frac{2}{n\pi} \Rightarrow \boxed{b_n = -\frac{2}{n\pi}}$$

$$\textcircled{1} \Rightarrow f(x) = \frac{3}{2} + \sum_{n=1,3,5,\dots} \frac{-4}{n^2\pi^2} \cos \frac{n\pi x}{2} + \sum_{n=1}^{\infty} \frac{-2}{n\pi} \sin \frac{n\pi x}{2}$$

$$= \frac{3}{2} - \frac{4}{\pi^2} \left[\frac{\cos \frac{\pi x}{2}}{1^2} + \frac{\cos \frac{3\pi x}{2}}{3^2} + \frac{\cos \frac{5\pi x}{2}}{5^2} + \dots \right] - \frac{2}{\pi} \left[\frac{\sin \frac{\pi x}{2}}{1} + \frac{\sin \pi x}{2} + \frac{\sin \frac{3\pi x}{2}}{3} + \dots \right]$$

5. find Expand $f(x) = e^{-x}$ as a fourier series in the interval $[-1, 1]$.

Sol. given $f(x) = e^{-x}$

$$\text{let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$\text{Here } c = -1, \quad c + 2l = 1 \Rightarrow -1 + 2l = 1 \Rightarrow 2l = 2 \Rightarrow \boxed{l = 1}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x \quad \text{--- (1)}$$

$$a_0 = \frac{1}{l} \int_{-1}^1 f(x) dx = \int_{-1}^1 e^{-x} dx = -(\bar{e}^x)_{-1}^1 = -(\bar{e}^1 - \bar{e}^{-1}) = e - \bar{e}^1$$

$$a_n = \frac{1}{l} \int_{-1}^1 f(x) \cos \frac{n\pi x}{l} dx = \int_{-1}^1 e^{-x} \cos n\pi x dx$$

$$\text{WKT } \int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} [a \cos bx + b \sin bx]$$

$$= \left(\frac{e^{-x}}{n^2\pi^2 + 1} [-\cos n\pi x + n\pi \sin n\pi x] \right)_{-1}^1$$

$$= \frac{e^{-1}}{n^2\pi^2 + 1} [-\cos n\pi + n\pi \sin n\pi] - \frac{e^1}{n^2\pi^2 + 1} [-\cos n\pi - n\pi \sin n\pi]$$

$$= \frac{e^{-1}}{n^2\pi^2 + 1} [(-1)^n] - \frac{e^1}{n^2\pi^2 + 1} [(-1)^n] = \frac{(-1)^{n+1} (e^{-1} - e^1)}{n^2\pi^2 + 1} = \frac{2(-1)^n (e - e^{-1})}{n^2\pi^2 + 1}$$

$$a_n = \frac{2(-1)^n \sinh h}{n^2 \pi^2 + 1}$$

$$b_n = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx = \int_{-1}^1 e^{-x} \sin n\pi x dx$$

$$\text{WKT } \int e^{ax} \sin bx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$$

$$= \left(\frac{e^{-x}}{n^2 \pi^2 + 1} [-\sin n\pi x - n\pi \cos n\pi x] \right)_{-1}^1$$

$$\begin{cases} \sin n\pi = 0 \\ \cos n\pi = (-1)^n \end{cases}$$

$$= \frac{e^{-1}}{n^2 \pi^2 + 1} [-\sin n\pi - n\pi \cos n\pi] - \frac{e^1}{n^2 \pi^2 + 1} [\sin n\pi - n\pi \cos n\pi]$$

$$= \frac{e^{-1}}{n^2 \pi^2 + 1} [-n\pi (-1)^n] - \frac{e^1}{n^2 \pi^2 + 1} [-n\pi (-1)^n]$$

$$= \frac{n\pi (-1)^n}{n^2 \pi^2 + 1} (e - e^{-1}) = \frac{2n\pi (-1)^n \sinh h}{n^2 \pi^2 + 1}$$

$$b_n = \frac{2n\pi (-1)^n \sinh h}{n^2 \pi^2 + 1}$$

$$\textcircled{1} \Rightarrow f(x) = \frac{1}{2} \sinh h + \sum_{n=1}^{\infty} \frac{2(-1)^n \sinh h}{n^2 \pi^2 + 1} \cos n\pi x + \sum_{n=1}^{\infty} \frac{2n\pi (-1)^n \sinh h}{n^2 \pi^2 + 1} \sin n\pi x$$

6. Find the fourier series for the function $f(x) = \pi x, 0 \leq x \leq 1$

$$= \pi(2-x), 1 \leq x \leq 2$$

sol. given $f(x) = \pi x, 0 \leq x \leq 1$
 $= \pi(2-x), 1 \leq x \leq 2$

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$\text{Here } (c=0, c+2l=2 \Rightarrow 2l=2 \Rightarrow l=1)$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x \quad \text{--- (1)}$$

$$a_0 = \frac{1}{l} \int_0^2 f(x) dx = \frac{1}{1} \left[\int_0^1 \pi x dx + \int_1^2 \pi(2-x) dx \right]$$

$$= \int_0^1 \frac{\pi}{2} (x^2)' + \pi \int_1^2 (2-x) dx = \frac{\pi}{2} + \pi \left[2(x) - \frac{1}{2} (x^2) \right]_1^2$$

$$= \frac{\pi}{2} + \pi \left[2 - \frac{1}{2} (3) \right] = \frac{\pi}{2} + 2\pi - \frac{3\pi}{2} = \frac{5\pi - 3\pi}{2} = \frac{2\pi}{2}$$

$$a_0 = \frac{5\pi - 3\pi}{2}$$

$$a_0 = \pi$$

$$\begin{aligned}
 a_n &= \frac{1}{l} \int_0^l f(x) \cos n\pi x \, dx = \int_0^2 f(x) \cos n\pi x \, dx \\
 &= \int_0^1 \pi x \cos n\pi x \, dx + \int_1^2 \pi(2-x) \cos n\pi x \, dx \\
 &= \pi \left[\left(x \frac{\sin n\pi x}{n\pi} \right)_0^1 - \int_0^1 \frac{\sin n\pi x}{n\pi} \, dx \right] + \pi \left[\left((2-x) \frac{\sin n\pi x}{n\pi} \right)_1^2 - \int_1^2 (-1) \frac{\sin n\pi x}{n\pi} \, dx \right] \\
 &= \pi \left[\frac{1}{n\pi} (\sin n\pi - 0) \right] - \frac{1}{n} \left[-\frac{\cos n\pi x}{n\pi} \right]_0^1 + \frac{\pi}{n\pi} [0 - \sin n\pi] \\
 &\quad - \frac{1}{n} \left(-\frac{\cos n\pi x}{n\pi} \right)_1^2 \\
 &= 0 + \frac{1}{n^2 \pi^2} (\cos n\pi - 1) + 0 + \frac{1}{n^2 \pi^2} (\cos 2n\pi - \cos n\pi) \\
 &= \frac{1}{n^2 \pi^2} [(-1)^n - 1] + \frac{1}{n^2 \pi^2} [1 - (-1)^n] = \frac{(-1)^n - 1 + 1 - (-1)^n}{n^2 \pi^2} = 0
 \end{aligned}$$

$$\boxed{a_n = 0}$$

$$\begin{cases} \sin n\pi = 0 \\ \cos n\pi = (-1)^n \\ \cos 2n\pi = 1 \end{cases}$$

$$\begin{aligned}
 b_n &= \frac{1}{l} \int_0^l f(x) \sin n\pi x \, dx = \int_0^2 f(x) \sin n\pi x \, dx \\
 &= \int_0^1 \pi x \sin n\pi x \, dx + \int_1^2 \pi(2-x) \sin n\pi x \, dx \\
 &= \pi \left[\left(x \left(-\frac{\cos n\pi x}{n\pi} \right) \right)_0^1 - \int_0^1 \left(-\frac{\cos n\pi x}{n\pi} \right) \, dx \right] \\
 &\quad + \pi \left[\left((2-x) \left(-\frac{\cos n\pi x}{n\pi} \right) \right)_1^2 - \int_1^2 (-1) \left(-\frac{\cos n\pi x}{n\pi} \right) \, dx \right] \\
 &= \pi \left[-\frac{1}{n\pi} (\cos n\pi - 0) \right] + \frac{1}{n} \left(\frac{\sin n\pi x}{n\pi} \right)_0^1 + \pi \left[-\frac{1}{n\pi} (0 - \cos n\pi) \right] \\
 &\quad - \frac{1}{n} \left(\frac{\sin n\pi x}{n\pi} \right)_1^2 \\
 &= -\frac{1}{n} (-1)^n + \frac{1}{n^2 \pi^2} (\sin n\pi - 0) + \frac{1}{n} (-1)^n - \frac{1}{n^2 \pi^2} (\sin 2n\pi - \sin n\pi) \\
 &= 0 - 0 = 0
 \end{aligned}$$

$$\Rightarrow \boxed{b_n = 0}$$

$$\textcircled{1} \Rightarrow f(x) = \frac{\pi}{2} + 0 + 0 = \frac{\pi}{2}$$

15/6/15 Even Fourier Series for Even & Odd Functions

Even Function: A function $f(x)$ is said to be even if $f(-x) = f(x)$ *x

Ex: x^2 , $\cos x$ are even functions of x .

Note 1) The graph of an even function is symmetrical about the

y-axis. 2). Modulus function is always an even function
i.e. $|x|$, $|\sin x|$, $|\cos x|$ etc are even functions.

Odd Function: A function $f(x)$ is said to be odd if $f(-x) = -f(x)$

Ex: x^3 , $\sin x$, x are even odd functions of x .

Note, The graph of an odd function is symmetrical about the origin.

Fourier Series for even & odd function:

→ If $f(x)$ is even in $(-\pi, \pi)$ then its fourier series is in the form of $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$, [$\therefore b_n = 0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$
- π (even) (odd)
= odd]

where $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$, $a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$

→ If $f(x)$ is odd in $(-\pi, \pi)$ then its fourier series is of the form $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$, where $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$

→ If $f(x)$ is even in $(-l, l)$ then its fourier series is in the form of $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$, [$\therefore a_0 = \frac{1}{l} \int_{-l}^l f(x) dx = 0$
 $a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx = 0$
- π (odd) (even) = odd,]
where $a_0 = \frac{2}{l} \int_0^l f(x) dx$, $a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx$.

→ If $f(x)$ is odd in $(-l, l)$ then its fourier series is of the form $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$, where $b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$

Formula. 1) If $f(x)$ is defined in $[-a, a]$ then
 $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$, when $f(x)$ is an even function
 $= 0$, when $f(x)$ is an odd function

1. Find the Fourier series to represent the function $f(x) = x^2 - 2$, $-2 \leq x \leq 2$.

Given $f(x) = x^2 - 2$

$$f(-x) = (-x)^2 - 2 = x^2 - 2 = f(x)$$

$\therefore f(x)$ is an even function & its Fourier series contains a_0 & a_n only.

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

$$\text{Here } l=2, \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2} \quad \text{--- (1)}$$

where

$$a_0 = \frac{2}{l} \int_0^l f(x) dx = \frac{2}{2} \int_0^2 (x^2 - 2) dx \quad (\because l=2)$$

$$= \frac{1}{3} (x^3)_0^2 - 2(x)_0^2 = \frac{8}{3} - 4 = -\frac{4}{3} \Rightarrow \boxed{a_0 = -4/3}$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx = \frac{2}{2} \int_0^2 (x^2 - 2) \cos \frac{n\pi x}{2} dx$$

$$= \left[(x^2 - 2) \frac{\sin \frac{n\pi x}{2}}{(\frac{n\pi}{2})} - 2x \left(\frac{-\cos \frac{n\pi x}{2}}{(\frac{n\pi}{2})^2} \right) + 2 \left(\frac{-\sin \frac{n\pi x}{2}}{(\frac{n\pi}{2})^3} \right) \right]_0^2$$

$$= \left[2 \sin n\pi \left(\frac{2}{n\pi} \right) + 4 \left(\frac{4}{n^2 \pi^2} \right) \cos n\pi - 2 \left(\frac{8}{n^3 \pi^3} \right) \sin n\pi \right] - [0 \text{ to } 0]$$

$$= \frac{4}{n\pi} (0) + \frac{16}{n^2 \pi^2} (-1)^n - \frac{16}{n^3 \pi^3} (0) \quad \left[\begin{array}{l} \because \sin n\pi = 0 \\ \cos n\pi = (-1)^n \end{array} \right]$$

$$\boxed{a_n = \frac{16}{n^2 \pi^2} (-1)^n}$$

$$\left[\int u v dx = u v_1 - u' v_2 + u'' v_3 - u''' v_4 + \dots \right]$$

where 'i' denotes diff & suffix denotes integration

$$\therefore f(x) = \left(-\frac{4}{3} \right) + \sum_{n=1}^{\infty} \frac{16}{n^2 \pi^2} (-1)^n \cos \frac{n\pi x}{2}$$

$$= -\frac{4}{3} + \sum_{n=1}^{\infty} \frac{16}{n^2 \pi^2} (-1)^n \cos \frac{n\pi x}{2}$$

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2. Obtain the Fourier series for $f(x) = |x|$ in $[-\pi, \pi]$, hence

show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

sol. given $f(x) = |x|$

$$f(-x) = |-x| = x = f(x)$$

$f(x)$ is an even function $\therefore$ its p.s contains a_0 & a_n only.

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{--- ①}$$

$$\text{where } a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} |x| dx = \frac{2}{\pi} \int_0^{\pi} x dx \quad \left(\because |x| = -x, x < 0 \right. \\ \left. = +x, x > 0 \right)$$

$$= \frac{2}{\pi} \left(\frac{1}{2} \right) (x^2)_0^{\pi} = \frac{1}{\pi} (\pi^2) = \pi \Rightarrow \boxed{a_0 = \pi}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi} |x| \cos nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \cos nx dx = \frac{2}{\pi} \left[x \frac{\sin nx}{n} - \left(-\frac{\cos nx}{n^2} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{\pi}{n} \sin n\pi + \frac{\cos n\pi}{n^2} - \left(0 + \frac{1}{n^2} \right) \right]$$

$$= \frac{2}{\pi} \left[0 + \frac{1}{n^2} (-1)^n - \frac{1}{n^2} \right] \Rightarrow a_n = \frac{2}{n^2 \pi} [(-1)^n - 1]$$

$$\text{①} \Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi} \left[\frac{1}{n^2} (-1)^n - \frac{1}{n^2} \right]$$

$$\therefore a_n = 0 \quad \text{if } n \text{ is even}$$

$$= \frac{-4}{n^2 \pi}, \text{ if } n \text{ is odd}$$

$$\text{②} \Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1,3,5,\dots}^{\infty} \frac{-4}{n^2 \pi} \cos nx$$

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right] \quad \text{--- ②}$$

Deduction: $f(x)$ is continuous at $x=0$ & hence fourier series converges to $f(0)$.

$$\text{②} \Rightarrow 0 = \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right] \Rightarrow \frac{\pi}{2} = \frac{4}{\pi} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

3. Expand $f(x) = 1 + \frac{2x}{\pi}$ if $-\pi \leq x \leq 0$ as a fourier series
 $= 1 - \frac{2x}{\pi}$ if $0 \leq x \leq \pi$

hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

sol. $f(-x) = 1 + \frac{2(-x)}{\pi} = 1 - \frac{2x}{\pi} = f(x)$ in $[-\pi, 0]$
 $= f(x)$ in $[0, \pi]$

$f(-x) = 1 - \frac{2(-x)}{\pi} = 1 + \frac{2x}{\pi}$ in $[0, \pi]$

$f(x) = f(x)$ in $[-\pi, 0]$

$f(x)$ is an even function in $[-\pi, \pi]$ $\therefore$ its F.S contains a_0 & a_n only.

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ — (1)

$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} (1 - \frac{2x}{\pi}) dx$
 $= \frac{2}{\pi} \left[(x)_0^{\pi} - \frac{2}{\pi} (x^2)_0^{\pi} \right] = \frac{2}{\pi} \left[\pi - \frac{\pi^2}{\pi} \right] = 0$
 $\Rightarrow a_0 = 0$

$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi} (1 - \frac{2x}{\pi}) \cos nx dx$

$= \frac{2}{\pi} \left[(1 - \frac{2x}{\pi}) \frac{\sin nx}{n} - \left(-\frac{2}{\pi} \right) \left(-\frac{\cos nx}{n^2} \right) \right]_0^{\pi}$

$= \frac{2}{\pi} \left[-\frac{\sin n\pi}{n} - \frac{2}{n^2\pi} \cos n\pi + \frac{2}{n^2\pi} \right]$

$= \frac{2}{\pi} \left[0 - \frac{2}{n^2\pi} (-1)^n + \frac{2}{n^2\pi} \right] = \frac{4}{n^2\pi^2} [1 - (-1)^n]$

$a_n = 0$, if n is even

$= \frac{8}{n^2\pi^2}$, if n is odd

(1) $\therefore f(x) = \sum_{n=1,3,5,\dots}^{\infty} \frac{8}{n^2\pi^2} \cos nx = \frac{8}{\pi^2} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right]$

$$= 1 - \frac{2x}{\pi} = \frac{8}{\pi^2} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right] \quad \text{--- ②}$$

Deduction $f(x)$ is continuous at $x=0$ & hence its F.S is converged to $f(0)$.

put $x=0$ in ②

$$1 = \frac{8}{\pi^2} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right] \Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

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4. Obtain Fourier series for the function $f(x)$ given by

$$f(x) = \frac{1}{2}(\pi+x) \text{ for } -\pi < x < 0$$

$$= \frac{1}{2}(\pi-x) \text{ for } 0 < x < \pi$$

$$\text{So, } f(-x) = \frac{1}{2}(\pi-x) \text{ in } (-\pi, 0) \\ = -f(x) \text{ in } (0, \pi) \quad \text{ie. } f(-x) = -f(x) \text{ in } (-\pi, \pi)$$

$$\therefore f(-x) = \frac{1}{2}(\pi+x) \text{ in } (0, \pi)$$

$$= -f(x) \text{ in } (-\pi, 0)$$

$\therefore f(x)$ is an odd function in $(-\pi, \pi)$ & its F.S contains b_n only.

$$\text{let } f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- ①}$$

$$\text{where } b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \frac{1}{2}(\pi-x) \sin nx \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} (\pi-x) \sin nx \, dx = \frac{1}{\pi} \left[(\pi-x) \left(-\frac{\cos nx}{n} \right) - (-1) \left(-\frac{\sin nx}{n^2} \right) \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[0 - \frac{\sin n\pi}{n^2} + \frac{\pi}{n} \cos n\pi + \frac{\sin 0}{n^2} \right] = \frac{1}{\pi} \left[\frac{\pi}{n} \right] = \frac{1}{n}$$

$$\boxed{b_n = \frac{1}{n}}$$

$$\text{①} \Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx$$

① Expand the function $f(x) = x^v$ as a Fourier series in $[-\pi, \pi]$ and hence deduce that

$$(i) \frac{1}{1^v} - \frac{1}{2^v} + \frac{1}{3^v} - \frac{1}{4^v} + \dots = \frac{\pi^v}{12}$$

$$(ii) \frac{1}{1^v} + \frac{1}{2^v} + \frac{1}{3^v} + \frac{1}{4^v} + \dots = \frac{\pi^v}{6}$$

$$(iii) \frac{1}{1^v} + \frac{1}{3^v} + \frac{1}{5^v} + \frac{1}{7^v} + \dots = \frac{\pi^v}{8}$$

Sol.
Given function $f(x) = x^v$ defined in $[-\pi, \pi]$
Clearly $f(x)$ is an even function in $[-\pi, \pi]$

'w.k.T' Fourier series expansion of an even function

in $[-\pi, \pi]$ is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ — (1)

where $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$

$$= \frac{2}{\pi} \int_0^{\pi} x^v dx = \frac{2}{\pi} \left(\frac{x^{v+1}}{v+1} \right)_0^{\pi}$$

$$\Rightarrow a_0 = \frac{2}{\pi} \left(\frac{\pi^{v+1}}{v+1} - 0 \right) = \frac{2\pi^v}{v}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \underset{u}{x^2} \underset{v}{\cos nx} \, dx$$

$$= \frac{2}{\pi} [uv_1 - uv_2 + u''v_3]$$

Here

$$u = x^2$$

$$v = \cos nx$$

$$u' = 2x$$

$$v_1 = \int v \, dx = \sin nx / n$$

$$u'' = 2$$

$$v_2 = \int v_1 \, dx = -\cos nx / n^2$$

$$v_3 = \int v_2 \, dx = -\sin nx / n^3$$

$$a_n = \frac{2}{\pi} \left[x^2 \left(\frac{\sin nx}{n} \right) - 2x \left(-\frac{\cos nx}{n^2} \right) + 2 \left(-\frac{\sin nx}{n^3} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{\pi^2 \sin n\pi}{n} + \frac{2\pi \cos n\pi}{n^2} - \frac{2 \sin n\pi}{n^3} - 0 \right]$$

$$= \frac{2}{\pi} \left[\frac{2\pi}{n^2} (-1)^n \right]$$

$$(\because \sin n\pi = 0 \\ \cos n\pi = (-1)^n \\ \sin 0 = 0)$$

$$\boxed{a_n = \frac{4}{n^2} (-1)^n}$$

Sub a_n values in (1), we have

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos nx \quad \text{--- (2)}$$

Deduction

(i) put $x=0$ in (2) we have

$$0 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} (-1)^n$$

$$\Rightarrow 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{3}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$$

(ii) put $x = \pi$ in (i) we have.

$$\pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos(n\pi)}{n^2}$$

$$4 \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{n^2} = \pi^2 - \frac{\pi^2}{3}$$

$$\Rightarrow 4 \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2} = \frac{2\pi^2}{3}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad (\because (-1)^{2n} = 1)$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6}$$

(iii) adding (i) & (ii) we have.

$$\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \right) + \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \right) =$$

$$\Rightarrow 2 \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right) = \frac{\frac{\pi^2}{12} + \frac{\pi^2}{6}}{\frac{\pi^2 + 2\pi^2}{12}}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{3\pi^2}{8 \times 12}$$

$$\Rightarrow \boxed{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}}$$

Half-Range Series: In the previous section, we have considered the Fourier series expansion of a function $f(x)$ defined in $(-l, l)$ of length $2l$ or $(-\pi, \pi)$ of length 2π . In various engineering problems, it is required to obtain a Fourier series of $f(x)$ defined in half of the above interval say $(0, l)$ or $(0, \pi)$. Such expansions or series is known as Half-Range Series.

As it is immaterial, whatever the function may be outside the range, $0 < x < l$. We extend the function to cover the range $-l < x < l$. So, the new function becomes even or odd. Therefore the Fourier series of such functions having cosine terms or sine terms only.

The expansion containing only cosine terms is known as "Half-range cosine series".

The expansion containing only sine terms is known as "Half-range sine series".

Half-range Cosine Series: The half-range cosine series of $f(x)$ in the interval $(0, l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}, \quad (0 < x < l)$$

$$\text{where } a_0 = \frac{2}{l} \int_0^l f(x) dx, \quad a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx$$

Half-range Sine Series: The half-range sine series of $f(x)$ in the interval $(0, l)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, \quad \text{where } b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

Note: It is required to find the half-range fourier cosine or sine series of $f(x)$ in $(0, \pi)$ by replacing 'l' with ' π ' in the above formulae

1. Find the half-range fourier sine series for $f(x) = x(\pi - x)$ in $(0, \pi)$ & hence deduce that $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \dots = \frac{\pi^3}{32}$

Ans. The half-range sine series of $f(x)$ in the interval $(0, \pi)$ is given by $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ — (1)

$$\text{where } b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x(\pi - x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2) \sin nx \, dx$$

$$= \frac{2}{\pi} \left[(\pi x - x^2) \left(-\frac{\cos nx}{n} \right) - (\pi - 2x) \left(-\frac{\sin nx}{n^2} \right) \right]_0^{\pi} + (-2) \left(\frac{\cos nx}{n^3} \right) \Big|_0^{\pi}$$

$$= \frac{2}{\pi} \left[0 + (-\pi) \frac{\sin n\pi}{n^2} - \frac{2}{n^3} \cos n\pi + 0 - \pi(0) + \frac{2}{n^3} \right]$$

$$= \frac{2}{\pi} \left[0 - \frac{n^2}{n^3} (-1)^n + \frac{2}{n^3} \right] = \frac{4}{n^3 \pi} [1 - (-1)^n]$$

$$b_n = 0, \text{ if } n \text{ is even}$$

$$= \frac{8}{n^3 \pi}, \text{ if } n \text{ is odd}$$

$$\text{①} \Rightarrow f(x) = \sum_{n=1,3,5,\dots}^{\infty} \frac{8}{n^3 \pi} \sin nx = \frac{8}{\pi} \left[\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \dots \right]$$

$$\therefore x(\pi - x) = \frac{8}{\pi} \left[\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \dots \right] \text{ — (2)}$$

Deduction: $f(x)$ is continuous at $x = \pi/2$ & hence it is converged to $f(\pi/2)$. put $x = \pi/2$ in (2), we get

$$\text{②} \Rightarrow \frac{\pi}{2} \left(\pi - \frac{\pi}{2} \right) = \frac{8}{\pi} \left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \dots \right]$$

$$\Rightarrow \frac{\pi^2}{4} = \frac{8}{\pi} \left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \dots \right]$$

$$\therefore \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \dots = \frac{\pi^3}{32}$$

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2. Find the half-range sine series for $f(x) = 1$, $0 < x < \pi/2$,

$$x < -1, \pi/2 < x < \pi$$

Ans. The half-range Fourier sine series for $f(x)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

$$\text{where } b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} f(x) \sin nx \, dx + \int_{\pi/2}^{\pi} f(x) \sin nx \, dx \right]$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} \sin nx \, dx - \int_{\pi/2}^{\pi} \sin nx \, dx \right]$$

$$= \frac{2}{\pi} \left[\left(-\frac{\cos nx}{n} \right)_0^{\pi/2} + \left(\frac{\cos nx}{n} \right)_{\pi/2}^{\pi} \right]$$

$$= \frac{2}{\pi} \left[-\frac{\cos n\pi/2}{n} + \frac{1}{n} + \frac{\cos n\pi}{n} - \frac{\cos n\pi/2}{n} \right]$$

$$= \frac{2}{\pi} \left[\frac{1}{n} + \frac{1}{n} (-1)^n - \frac{2 \cos n\pi/2}{n} \right]$$

$$b_n = \frac{2}{n\pi} [1 + (-1)^n - 2 \cos n\pi/2]$$

$$\text{①} \Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{1}{n} [1 + (-1)^n - 2 \cos n\pi/2] \sin nx$$

3. Find the half-range Fourier sine series of $f(x) = e^x$ in $(0, 1)$.

Ans. The half range Fourier sine series for $f(x)$ is given

$$\text{by } f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \text{--- (1)} \quad \text{Here } l=1$$

$$\text{where } b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{2}{1} \int_0^1 e^x \sin n\pi x \, dx = 2 \left[s \right]$$

WKT

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$$

$$= 2 \int_0^1 \left[\frac{e^x}{n^2\pi^2 + 1} (\sin n\pi x - n\pi \cos n\pi x) \right]_0^1 \quad \begin{matrix} a=1 \\ b=n\pi \end{matrix}$$

$$= 2 \left[\frac{e}{n^2\pi^2 + 1} (\sin n\pi - n\pi \cos n\pi) - \frac{1}{n^2\pi^2 + 1} (0 - n\pi) \right]$$

$$b_n = 2 \left[\frac{e}{n^2\pi^2 + 1} (-n\pi (-1)^n) + \frac{n\pi}{n^2\pi^2 + 1} \right]$$

$$b_n = \frac{2n\pi}{1 + n^2\pi^2} [1 - e(-1)^n]$$

$$\Rightarrow f(x) = 2\pi \sum_{n=1}^{\infty} \frac{n}{1 + n^2\pi^2} (1 - e(-1)^n) \sin n\pi x$$

4. Find the half-range Fourier cosine series of $f(x) = x$ in $0 < x < \pi$.

Ans. The half-range Fourier cosine series for $f(x)$ in $(0, \pi)$ is

$$\text{given by } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{--- (1)}$$

$$\text{where } a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) \, dx = \frac{2}{\pi} \int_0^{\pi} x \, dx = \frac{2}{\pi} \left(\frac{x^2}{2} \right)_0^{\pi} \\ = \frac{1}{\pi} (\pi^2 - 0) = \pi \Rightarrow \boxed{a_0 = \pi}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx = \frac{2}{\pi} \int_0^{\pi} x \cos nx \, dx$$

$$= \frac{2}{\pi} \left[x \frac{\sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^{\pi} = \frac{2}{\pi} \left[\frac{\pi \sin n\pi}{n} + \frac{\cos n\pi}{n^2} - \frac{1}{n^2} \right]$$

$$= \frac{2}{\pi} \left[\frac{1}{n^2} ((-1)^n - 1) \right] = \frac{2}{n^2\pi} [(-1)^n - 1]$$

$\therefore a_n = 0$, if n is even

$= \frac{-4}{n^2\pi}$, if n is odd

$$\textcircled{1} \Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1,3,5,\dots}^{\infty} \frac{-4}{n^2 \pi} \cos nx = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{\cos nx}{n^2}$$

5. Obtain fourier cosine series for $f(x) = x \sin x$, $0 < x < \pi$

Ans) The half-range fourier cosine series for $f(x)$ in $(0, \pi)$

is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ — (1)

where

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} x \sin x dx \\ &= \frac{2}{\pi} \left[x(-\cos x) + \sin x \right]_0^{\pi} = \frac{2}{\pi} [-\pi \cos \pi + \sin \pi + 0] \\ &= \frac{2}{\pi} [\pi] = 2 \Rightarrow \boxed{a_0 = 2} \end{aligned}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi} x \sin x \cos nx dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x (2 \sin nx \sin x) dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x [\sin (nx+x) - \sin (nx-x)] dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x \sin (n+1)x dx - \frac{1}{\pi} \int_0^{\pi} x \sin (n-1)x dx$$

$$= \frac{1}{\pi} \left[x \left(\frac{-\cos (n+1)x}{n+1} \right) + \frac{\sin (n+1)x}{(n+1)^2} \right]_0^{\pi} - \frac{1}{\pi} \left[x \left(\frac{-\cos (n-1)x}{n-1} \right) + \frac{\sin (n-1)x}{(n-1)^2} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{-\pi \cos (n+1)\pi}{n+1} + \frac{\sin (n+1)\pi}{(n+1)^2} - (0+0) \right] - \frac{1}{\pi} \left[\frac{-\pi \cos (n-1)\pi}{n-1} + \frac{\sin (n-1)\pi}{(n-1)^2} - (0+0) \right]$$

$$= \frac{-1}{n+1} \cos (n+1)\pi + \frac{\sin (n+1)\pi}{\pi (n+1)^2} + \frac{1}{n-1} \cos (n-1)\pi - \frac{\sin (n-1)\pi}{\pi (n-1)^2}$$

$$= \frac{-1}{n+1} (-1)^{n+1} + \frac{1}{n-1} (-1)^{n-1} = \frac{+1}{n+1} (-1)^n + \frac{1}{n-1} (-1)^n$$

$$= (-1)^n \left[\frac{1}{n+1} + \frac{1}{n-1} \right] = (-1)^n \left(\frac{n-1+n+1}{n^2-1} \right) = \frac{2(-1)^n}{n^2-1}, \text{ if } n \neq 1$$

if $n=1$, then the above integral becomes

$$a_1 = \frac{2}{\pi} \int_0^{\pi} f(x) \cos x \, dx = \frac{2}{\pi} \int_0^{\pi} x \sin x \cos x \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x \sin 2x \, dx = \frac{1}{\pi} \int_0^{\pi} x \sin 2x \, dx$$

$$= \frac{1}{\pi} \left[\frac{x}{2} (-\cos 2x) + \frac{\sin 2x}{4} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[-\frac{\pi}{2} \cos 2\pi + \frac{\sin 2\pi}{4} \right] = -\frac{1}{2} (1) = -\frac{1}{2} \Rightarrow \boxed{a_1 = -\frac{1}{2}}$$

$$\textcircled{1} \Rightarrow f(x) = 1 - \frac{1}{2} \cos x + \sum_{n=2}^{\infty} \frac{2}{n^2-1} (-1)^{n+1} \cos nx$$

1. Expand the function $f(x) = \frac{\pi^x}{12} - \frac{x^x}{4}$ in fourier series in the interval $(-\pi, \pi)$.
2. Find the F.S. of $f(x) = x^3$ in $(-\pi, \pi)$.
3. Find the F.S. of $f(x) = -K$, if $-\pi < x < 0$
 $= K$, if $0 < x < \pi$.
4. Find the F.S. of $f(x) = |\sin x|$ in $(-\pi, \pi)$
5. prove that the interval $(-\pi, \pi)$,

$$x \cos x = -\frac{1}{2} \sin x + 2 \sum_{n=2}^{\infty} \frac{n(-1)^n}{n^2-1} \sin nx.$$
6. Obtain the F.S. for $\sqrt{1-\cos x}$ in the interval $-\pi \leq x \leq \pi$.
7. Find the Half range cosine series for the function
 $f(x) = (x-1)^x, 0 \leq x \leq 1$. Hence deduce that

$$\frac{1}{1^x} + \frac{1}{2^x} + \frac{1}{3^x} + \dots = \frac{\pi^x}{6}.$$
8. Find the Half range sine series of the function

$$f(x) = \frac{1}{4} - x, 0 < x < \frac{1}{2}$$

$$= x - \frac{3}{4}, \frac{1}{2} < x < 1$$

Fourier Transforms

Fourier integral: A periodic function $f(x)$ defined in a finite interval $(-l, l)$ can be expressed in Fourier series. By extending this concept, non-periodic functions defined in $-\infty < x < \infty$ ($\forall x$) can be expressed as a Fourier integral.

Fourier integral theorem:

st: If $f(x)$ satisfies Dirichlet's conditions in every interval $(-l, l)$ and $f(x)$ is absolutely integrable in the interval $(-\infty, \infty)$ then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \alpha(t-x) dt d\alpha \quad \left[\begin{array}{l} \text{we will take} \\ \alpha = \lambda = p \end{array} \right]$$

$$\text{or } f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \alpha(t-x) dt d\alpha \quad \left[\begin{array}{l} \because \cos \alpha(t-x) \cos \alpha(t-x) \text{ is} \\ \text{an even function of } \alpha \text{ then} \\ \int_{-\infty}^{\infty} \cos \alpha(t-x) d\alpha = 2 \int_0^{\infty} \cos \alpha(t-x) d\alpha \end{array} \right]$$

which is valid at all points of continuity

Fourier sine and cosine integrals

$$\text{w.k.t. } \cos \alpha(t-x) = \cos \alpha t \cos \alpha x + \sin \alpha t \sin \alpha x \quad \left[\begin{array}{l} \cos(A-B) \\ = \cos A \cos B + \sin A \sin B \end{array} \right]$$

$\therefore$ Fourier integral of $f(x)$ can be written as

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) [\cos \alpha t \cos \alpha x + \sin \alpha t \sin \alpha x] dt d\alpha$$
$$= \frac{1}{\pi} \int_0^{\infty} \cos \alpha x \left[\int_{-\infty}^{\infty} f(t) \cos \alpha t dt \right] d\alpha + \frac{1}{\pi} \int_0^{\infty} \sin \alpha x \left[\int_{-\infty}^{\infty} f(t) \sin \alpha t dt \right] d\alpha \quad \text{--- (1)}$$

case(i): - when $f(t)$ is an odd function, $f(t) \cos \alpha t$ is odd while $f(t) \sin \alpha t$ is even.

The 1st integral in (1) is zero and, we get $\left[\int_{-a}^a f(x) dx = 0 \text{ when } f(x) \text{ is odd} \right]$

$$\therefore f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \alpha x \left[\int_0^{\infty} f(t) \sin \alpha t dt \right] d\alpha$$

This is called Fourier sine integral.

case(ii): when $f(t)$ is an even function, $f(t) \cos \alpha t$ is even, while $f(t) \sin \alpha t$ is odd.

The second integral in (1) is zero and, we get

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos \alpha x \left[\int_0^{\infty} f(t) \cos \alpha t dt \right] d\alpha$$

This is called Fourier cosine integral.

Note f.w.I: - At a point of discontinuity x_0 , the Fourier integral

$$= \frac{1}{2} [f(x_0^-) + f(x_0^+)] \text{ i.e. average of the left and right hand limits.}$$

1). Express the function $f(x) = 1$ for $|x| \leq 1$
 $= 0$ for $|x| > 1$ as a Fourier integral

and hence evaluate $\int_0^{\infty} \frac{\sin \alpha \cos \alpha x}{\alpha} d\alpha$.

Sol: - The Fourier integral of $f(x)$ is

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos \alpha (t-x) dt d\alpha$$

Let us take $f(t) = 1$, for $|t| \leq 1$
 $= 0$ for $|t| > 1$

$$\left[\begin{array}{l} \because |t| \leq 1, -1 \leq t \leq 1 \\ |t| > 1, t > 1 \text{ or } -t > 1 \\ \text{ie. } t > 1, t < -1 \end{array} \right]$$

$$\therefore f(x) = \frac{1}{\pi} \int_0^{\infty} \left[\int_{-\infty}^{-1} 0 dt + \int_{-1}^1 (1) \cos \alpha (t-x) dt + \int_1^{\infty} (0) dt \right] d\alpha \quad \left[\begin{array}{l} \int \cos(ax+b) dx \\ = \frac{\sin(ax+b)}{a} \end{array} \right]$$

$$= \frac{1}{\pi} \int_0^{\infty} \left[\frac{\sin \alpha (t-x)}{\alpha} \right]_{-1}^1 d\alpha$$

$$= \frac{1}{\pi} \int_0^{\infty} \frac{1}{\alpha} [\sin \alpha (1-x) - \sin \alpha (-1-x)] d\alpha$$

$$= \frac{1}{\pi} \int_0^{\infty} \frac{1}{\alpha} [\sin(\alpha - \alpha x) + \sin(\alpha + \alpha x)] d\alpha \quad \left[\begin{array}{l} \sin(A+B) + \sin(A-B) \\ = 2 \sin A \cos B \end{array} \right]$$

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \frac{1}{\alpha} 2 \sin \alpha \cos \alpha x d\alpha, \text{ is the required Fourier integral} \\ \text{--- (1)}$$

To find $\int_0^{\infty} \frac{\sin \alpha \cos \alpha x}{\alpha} d\alpha$: From (1), we have

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \frac{2 \sin \alpha \cos \alpha x}{\alpha} d\alpha$$

$$\text{ie. } f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \alpha \cos \alpha x}{\alpha} d\alpha$$

$$\frac{\pi}{2} f(x) = \int_0^{\infty} \frac{\sin \alpha \cos \alpha x}{\alpha} d\alpha$$

$$\therefore \int_0^{\infty} \frac{\sin \alpha \cos \alpha x}{\alpha} d\alpha = \frac{\pi}{2} f(x) = \frac{\pi}{2}, \quad |x| \leq 1 \\ = 0, \quad |x| > 1.$$

Q) Using Fourier integral representation, show that

$$\int_0^{\infty} \frac{\cos \lambda x + \alpha \sin \lambda x}{1 + \alpha^2} d\alpha = \begin{cases} 0, & \text{if } x < 0 \\ \frac{\pi}{2}, & \text{if } x = 0 \\ \pi e^{-x}, & \text{if } x > 0 \end{cases}$$

Sol:- The Fourier integral of $f(x)$ is

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \left[\int_{-\infty}^{\infty} f(t) \cos \alpha(t-x) dt \right] d\alpha$$

$$\text{Let us take } f(x) = 0, \text{ if } x < 0 \\ = \pi e^{-x}, \text{ if } x > 0$$

$$(\text{or}) f(t) = 0, \text{ if } t < 0 \\ = \pi e^{-t} \text{ if } t > 0$$

$$\begin{aligned} \therefore f(x) &= \frac{1}{\pi} \int_0^{\infty} \left[\int_{-\infty}^0 0 \cdot \cos \alpha(t-x) dt + \int_0^{\infty} \pi e^{-t} \cos \alpha(t-x) dt \right] d\alpha \\ &= \int_0^{\infty} \left[\int_0^{\infty} e^{-t} \cos \alpha(t-x) dt \right] d\alpha \left[\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} [a \cos bx + b \sin bx] \right] \\ &= \int_0^{\infty} \left[\frac{e^{-t}}{1 + \alpha^2} \left[-\cos \alpha(t-x) + \alpha \sin \alpha(t-x) \right] \right]_0^{\infty} d\alpha \\ &= \int_0^{\infty} \left[0 - \frac{e^{-0}}{1 + \alpha^2} (-\cos \alpha x + \alpha \sin \alpha(0-x)) \right] d\alpha \\ &= \int_0^{\infty} \left[-\frac{1}{1 + \alpha^2} (-\cos \alpha x - \alpha \sin \alpha x) \right] d\alpha \quad [\sin(-\theta) = -\sin \theta] \\ f(x) &= \int_0^{\infty} \frac{\cos \alpha x + \alpha \sin \alpha x}{1 + \alpha^2} d\alpha \end{aligned}$$

$$\text{At } x=0 \Rightarrow \int_0^{\infty} \frac{\cos \alpha x + \alpha \sin \alpha x}{1 + \alpha^2} d\alpha = 0 \text{ if } x < 0 \\ = \pi e^{-x}, \text{ if } x > 0$$

At $x=0$, $f(x)$ has a discontinuity. So $f(x) = \frac{1}{2} [f(0^+) + f(0^-)]$

$$\text{i.e. } f(x) = \frac{1}{2} [\pi + 0] = \frac{\pi}{2}$$

$$\therefore \int_0^{\infty} \frac{\cos \alpha x + \alpha \sin \alpha x}{1 + \alpha^2} d\alpha = \frac{\pi}{2}, \text{ if } x = 0.$$

$$\therefore \int_0^{\infty} \frac{\cos \alpha x + \alpha \sin \alpha x}{1 + \alpha^2} d\alpha = 0, \text{ if } x < 0 \\ = \frac{\pi}{2}, \text{ if } x = 0 \\ = \pi e^{-x}, \text{ if } x > 0.$$

Find the Fourier integral representation for the following functions

$$\begin{aligned} \text{i) } f(x) &= \frac{\pi}{2} \cos x, \quad |x| \leq \pi \\ &= 0, \quad |x| > \pi \end{aligned}$$

$$\begin{aligned} \text{ii) } f(x) &= 1, \quad |x| \leq a \\ &= 0, \quad |x| > a. \end{aligned}$$

$$\text{iii) } f(x) = e^{-|x|}, \quad -\infty < x < \infty$$

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 2. using Fourier cosine integral formula, prove that

$$e^{-ax} = \frac{2a}{\pi} \int_0^{\infty} \frac{\cos \alpha x}{a^2 + \alpha^2} d\alpha, \quad a > 0, x \geq 0$$

 Sol:- The Fourier cosine integral of $f(x)$ is

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos \alpha x \left[\int_0^{\infty} f(t) \cos \alpha t dt \right] d\alpha$$

let us take $f(t) = e^{-at}$ then

$$\begin{aligned} f(x) &= \frac{2}{\pi} \int_0^{\infty} \cos \alpha x \left[\int_0^{\infty} e^{-at} \cos \alpha t dt \right] d\alpha \\ &= \frac{2}{\pi} \int_0^{\infty} \cos \alpha x \left[\frac{e^{-at}}{a^2 + \alpha^2} (a \cos \alpha t + \alpha \sin \alpha t) \right]_0^{\infty} d\alpha \end{aligned}$$

$$= \frac{2}{\pi} \int_0^{\infty} \cos \alpha x \left[0 - \frac{1}{a^2 + \alpha^2} (-a(1) + \alpha(0)) \right] d\alpha$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{a \cos \alpha x}{a^2 + \alpha^2} d\alpha$$

$$e^{-ax} = \frac{2a}{\pi} \int_0^{\infty} \frac{\cos \alpha x}{a^2 + \alpha^2} d\alpha$$

3. using fourier integral formula, prove that

$$\int_0^{\infty} \frac{(x^2+2) \cos x}{x^4+4} dx = \frac{\pi}{2} e^{-x} \cos x.$$

Sol: since the integrant on LHS contains cosine terms, we use fourier cosine integral formula.

$\therefore$ By fourier cosine integral formula

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos \alpha x \left[\int_0^{\infty} f(t) \cos \alpha t dt \right] d\alpha$$

let us take $f(t) = e^{-t} \cos t$ then

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos \alpha x \left[\int_0^{\infty} e^{-t} \cos t \cos \alpha t dt \right] d\alpha$$

$$= \frac{1}{\pi} \int_0^{\infty} \cos \alpha x \left[\int_0^{\infty} e^{-t} (2 \cos t \cos \alpha t) dt \right] d\alpha$$

$$= \frac{1}{\pi} \int_0^{\infty} \cos \alpha x \left[\int_0^{\infty} e^{-t} (\cos(\alpha+1)t + \cos(\alpha-1)t) dt \right] d\alpha$$

$$= \frac{1}{\pi} \int_0^{\infty} \cos \alpha x \left[\int_0^{\infty} e^{-t} \cos(\alpha+1)t dt + \int_0^{\infty} e^{-t} \cos(\alpha-1)t dt \right] d\alpha$$

$$= \frac{1}{\pi} \int_0^{\infty} \cos \alpha x \left[\frac{1}{1^2 + (\alpha+1)^2} + \frac{1}{1^2 + (\alpha-1)^2} \right] d\alpha$$

$$= \frac{1}{\pi} \int_0^{\infty} \cos \alpha x \left[\frac{1}{\alpha^2 + 2\alpha + 2} + \frac{1}{\alpha^2 - 2\alpha + 2} \right] d\alpha$$

$$= \frac{1}{\pi} \int_0^{\infty} \cos \alpha x \left[\frac{2\alpha^2 + 4}{(\alpha^2 + 2)^2 - 4\alpha^2} \right] d\alpha$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos \alpha x \frac{(\alpha^2+2)}{\alpha^4+4} d\alpha$$

$$\therefore e^{-x} \cos x = \frac{2}{\pi} \int_0^{\infty} \cos \alpha x \frac{(\alpha^2+2)}{\alpha^4+4} d\alpha$$

$$\therefore \int_0^{\infty} \cos \alpha x \frac{(\alpha^2+2)}{(\alpha^4+4)} d\alpha = \frac{\pi}{2} e^{-x} \cos x.$$

4. using fourier sine integral formula, prove that

$$\int_0^{\infty} \frac{x \sin \alpha x}{a^2 + x^2} dx = \frac{\pi}{2} e^{-ax}, \quad a > 0, x > 0.$$

Sol: The fourier sine integral formula of $f(x)$ is

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \alpha x \left[\int_0^{\infty} f(t) \sin \alpha t dt \right] d\alpha$$

let us consider $f(t) = e^{-at}$ then

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \alpha x \left[\int_0^{\infty} e^{-at} \sin \alpha t dt \right] d\alpha$$

$$e^{-ax} = \frac{2}{\pi} \int_0^{\infty} \sin \alpha x \frac{\alpha}{a^2 + \alpha^2} d\alpha$$

$$e^{-ax} \cdot \frac{\pi}{2} = \int_0^{\infty} \frac{\alpha \sin \alpha x}{a^2 + \alpha^2} d\alpha$$

5. Using Fourier sine integral formula, prove that

$$e^{-ax} - e^{-bx} = \frac{2(b^2 - a^2)}{\pi} \int_0^{\infty} \frac{\alpha \sin \alpha x}{(\alpha^2 + a^2)(\alpha^2 + b^2)} d\alpha \quad a, b > 0$$

Sol:- The Fourier sine integral formula for $f(x)$ is

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \alpha x \left[\int_0^{\infty} f(t) \sin \alpha t dt \right] d\alpha$$

let $f(t) = e^{-at} - e^{-bt}$ then

$$= \frac{2}{\pi} \int_0^{\infty} \sin \alpha x \left[\int_0^{\infty} (e^{-at} - e^{-bt}) \sin \alpha t dt \right] d\alpha$$

$$= \frac{2}{\pi} \int_0^{\infty} \sin \alpha x \left[\int_0^{\infty} e^{-at} \sin \alpha t dt - \int_0^{\infty} e^{-bt} \sin \alpha t dt \right] d\alpha$$

$$= \frac{2}{\pi} \int_0^{\infty} \sin \alpha x \left[\frac{\alpha}{a^2 + \alpha^2} - \frac{\alpha}{b^2 + \alpha^2} \right] d\alpha$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \alpha \sin \alpha x \left[\frac{b^2 + \alpha^2 - a^2 - \alpha^2}{(a^2 + \alpha^2)(b^2 + \alpha^2)} \right] d\alpha$$

$$e^{-ax} - e^{-bx} = \frac{2}{\pi} (b^2 - a^2) \int_0^{\infty} \frac{\alpha \sin \alpha x}{(\alpha^2 + a^2)(\alpha^2 + b^2)} d\alpha$$

1. Express $f(x) = 1$ for $0 \leq x \leq \pi$ as a Fourier sine integral and
 $= 0$ for $x > \pi$

hence evaluate $\int_0^{\infty} \frac{1 - \cos \pi x}{x} \sin x \, dx$.

2. Using Fourier cosine integral, prove that $\int_0^{\infty} \frac{\cos x}{1+x^2} dx = \frac{\pi}{2} e^{-x}$, $x \geq 0$.

Complex (or) Exponential form of Fourier integral

The complex form of Fourier integral is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{i\alpha(t-x)} dt d\alpha$$

Proof:- By Fourier integral formula, $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \alpha(t-x) dt d\alpha$ — (1)

Here $\cos \alpha(t-x)$ is an even function of α .

Since $\sin \alpha(t-x)$ is an odd function of α $\left[\int_{-\infty}^{\infty} \sin \alpha(t-x) d\alpha = 0 \right]$

$$0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \sin \alpha(t-x) dt d\alpha \text{ — (2)}$$

① + i(2), we get

$$\text{i.e. } f(x) + i(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t) \cos \alpha(t-x) dt \right] d\alpha + i \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t) \sin \alpha(t-x) dt \right] d\alpha$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t) [\cos \alpha(t-x) + i \sin \alpha(t-x)] dt \right] d\alpha$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{i\alpha(t-x)} dt d\alpha \quad [\cos \theta + i \sin \theta = e^{i\theta}] \text{ — (3)}$$

which is the required complex form.

③ can be written as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{i\alpha t} e^{-i\alpha x} dt d\alpha$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t) e^{i\alpha t} dt \right] e^{-i\alpha x} d\alpha \text{ — I}$$

Infinite Fourier Transform

Denote the value of inner integral (I) by $F(\alpha)$

$$\text{i.e. } f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha) e^{-i\alpha x} d\alpha \text{ — (4)}$$

where $\left[F(\alpha) = \int_{-\infty}^{\infty} f(t) e^{i\alpha t} dt \right]$ is called Fourier transform

of $f(x)$ and is denoted by $F[f(x)]$

And (4) is called Inverse Fourier transform of $F(\alpha)$

$$\text{i.e. } \mathcal{F}^{-1}[F(\alpha)] = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha) e^{i\alpha x} d\alpha.$$

Fourier sine and cosine transforms

Fourier sine transform: By Fourier sine integral formula

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \alpha x \left[\int_0^{\infty} f(t) \sin \alpha t dt \right] d\alpha$$

Denote the value of inner integral by $F_s(\alpha)$

$$\text{i.e. } \boxed{f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \alpha x F_s(\alpha) d\alpha} \quad \text{--- (1)}$$

where $\boxed{F_s(\alpha) = \int_0^{\infty} f(t) \sin \alpha t dt}$ is called Fourier sine transform of $f(x)$ and is denoted by $f_s[f(x)]$. and (1) is called Inverse Fourier sine transform of $F_s(\alpha)$. $\left[\because F_s(\alpha) = F_s[f(x)] \right]$

Fourier cosine transform: By Fourier cosine integral formula

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos \alpha x \left[\int_0^{\infty} f(t) \cos \alpha t dt \right] d\alpha$$

Denote the value of inner integral by $F_c(\alpha)$

$$\text{i.e. } \boxed{f(x) = \frac{2}{\pi} \int_0^{\infty} \cos \alpha x F_c(\alpha) d\alpha} \quad \text{--- (2)}$$

where $\boxed{F_c(\alpha) = \int_0^{\infty} f(t) \cos \alpha t dt}$ is called

Fourier cosine transform, and (2) is called $\boxed{F_c(x) = F_c[f(x)]}$
Inverse Fourier cosine transform of $F_c(\alpha)$.

Properties:

$$1) f_s[x f(x)] = -\frac{d}{d\alpha} [F_c(\alpha)]$$

$$2) f_c[x f(x)] = \frac{d}{d\alpha} [F_s(\alpha)]$$

$$3) f\left[\frac{d^n}{dx^n} f(x)\right] = (-i\alpha)^n F(\alpha)$$

$$4) f_s[f'(x)] = -\alpha F_c(\alpha)$$

$$5) f_c[f'(x)] = \alpha F_s(\alpha) - f(0)$$

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Properties of Fourier transforms :-

1. Linearity property :-

st:- If $F(x)$ & $G(x)$ be the Fourier transforms of $f(x)$ & $g(x)$ respectively then

$$F[af(x) + bg(x)] = aF(x) + bG(x)$$

Proof: By the definition

$$F(x) = \int_{-\infty}^{\infty} e^{ixt} f(t) dt$$

$$G(x) = \int_{-\infty}^{\infty} e^{ixt} g(t) dt$$

$$\begin{aligned} \therefore F[af(x) + bg(x)] &= \int_{-\infty}^{\infty} e^{i\alpha x} [af(x) + bg(x)] dx \\ &= a \int_{-\infty}^{\infty} e^{i\alpha x} f(x) dx + b \int_{-\infty}^{\infty} e^{i\alpha x} g(x) dx \\ &= a F(\alpha) + b G(\alpha) \end{aligned}$$

2. change of scale properly:

St: If $F(\alpha)$ is the Fourier transform of $f(x)$
then $F[f(ax)] = \frac{1}{a} F\left(\frac{\alpha}{a}\right)$ where $a > 0$

Proof: By definition

$$\begin{aligned} F(\alpha) &= \int_{-\infty}^{\infty} e^{i\alpha x} f(x) dx \\ \therefore F[f(ax)] &= \int_{-\infty}^{\infty} e^{i\alpha x} f(ax) dx \end{aligned}$$

let $ax = t \Rightarrow x = \frac{t}{a}$
 $\infty dx = \frac{dt}{a}$
 if $x = -\infty \Rightarrow t = -\infty$
 $x = \infty \Rightarrow t = \infty$

$$F[f(ax)] = \int_{-\infty}^{\infty} e^{i\alpha \left(\frac{t}{a}\right)} f(t) \frac{dt}{a}$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} e^{i\alpha \left(\frac{t}{a}\right)} f(t) dt$$

$$F[f(ax)] = \frac{1}{a} F\left(\frac{\alpha}{a}\right)$$

Note:

$$\begin{aligned} 1. \quad F_s[af(x) + bg(x)] &= a F_s(\alpha) + b G_s(\alpha) \\ F_c[af(x) + bg(x)] &= a F_c(\alpha) + b G_c(\alpha) \end{aligned}$$

$$2. \quad F_s[f(ax)] = \frac{1}{a} F_s\left(\frac{\alpha}{a}\right)$$

$$F_c[f(ax)] = \frac{1}{a} F_c\left(\frac{\alpha}{a}\right)$$

3. shifting property:-

If $F(\alpha)$ is the Fourier transform of $f(x)$ then

$$F[f(x-a)] = e^{i\alpha a} F(\alpha)$$

Proof: By the definition

$$F(\alpha) = \int_{-\infty}^{\infty} e^{i\alpha x} f(x) dx$$

$$F[f(x-a)] = \int_{-\infty}^{\infty} e^{i\alpha x} f(x-a) dx$$

$$= \int_{-\infty}^{\infty} e^{i\alpha(a+t)} f(t) dt$$

$$= \int_{-\infty}^{\infty} e^{i\alpha a} e^{i\alpha t} f(t) dt$$

let $x = a + t$
 $z = a + t$
 $dz = dt$
 at $x = -\infty \Rightarrow t = -\infty$
 $z = \infty \Rightarrow t = \infty$

$$= \int_{-\infty}^{\infty} e^{i\alpha a} f(x) e^{i\alpha t} f(t) dt \quad [\because \text{by definition}]$$

$$\therefore F[f(x-a)] = e^{i\alpha a} F(\alpha)$$

Modulation theorem:-

st: If $F(\alpha)$ is the Fourier transform of $f(x)$
 then $F[f(x) \cos ax] = \frac{1}{2} [F(\alpha+a) + F(\alpha-a)]$
 $F(\alpha) = F[f(x)] = \int_{-\infty}^{\infty} e^{i\alpha x} f(x) dx$ - ①

Proof: By the definition

$$F[f(x) \cos ax] = \int_{-\infty}^{\infty} e^{i\alpha x} f(x) \cos ax dx$$

$$= \int_{-\infty}^{\infty} e^{i\alpha x} f(x) \left(\frac{e^{iax} + e^{-iax}}{2} \right) dx$$

$$= \frac{1}{2} \left[\int_{-\infty}^{\infty} f(x) e^{i\alpha x} e^{iax} dx + \int_{-\infty}^{\infty} f(x) e^{i\alpha x} e^{-iax} dx \right]$$

$$= \frac{1}{2} \left[\int_{-\infty}^{\infty} f(x) e^{i(\alpha+a)x} dx + \int_{-\infty}^{\infty} f(x) e^{i(\alpha-a)x} dx \right]$$

$$F[f(x) \cos ax] = \frac{1}{2} [F(\alpha+a) + F(\alpha-a)]$$

$$\text{||y } F[f(x) \sin ax] = \frac{1}{2i} [F(\alpha+a) - F(\alpha-a)]$$

Note:-

1. If $F_s(\alpha)$ be the Fourier sine transform of $f(x)$
 then $F_s[f(x) \cos ax] = \frac{1}{2} [F_s(\alpha+a) + F_s(\alpha-a)]$

$$F_s[f(x) \sin ax] = \frac{1}{2} [F_s(\alpha-a) - F_s(\alpha+a)]$$

2. If $F_c(\alpha)$ be the Fourier cosine transform of $f(x)$
 then $F_c[f(x) \cos ax] = \frac{1}{2} [F_c(\alpha+a) + F_c(\alpha-a)]$

$$F_c[f(x) \sin ax] = \frac{1}{2} [F_c(\alpha+a) - F_c(\alpha-a)]$$

1. Prove that $\mathcal{F}[x^n \cdot f(x)] = (-i)^n \frac{d^n}{d\alpha^n} [F(\alpha)]$
 By the definition of Fourier transform

$$F(\alpha) = \int_{-\infty}^{\infty} e^{i\alpha x} f(x) dx \quad \text{--- (1)}$$

$$\frac{d}{d\alpha} [F(\alpha)] = \int_{-\infty}^{\infty} \frac{d}{d\alpha} (e^{i\alpha x}) f(x) dx$$

$$= \int_{-\infty}^{\infty} (ix) e^{i\alpha x} f(x) dx \quad \text{--- (2)}$$

$$\frac{d^2}{d\alpha^2} [F(\alpha)] = \frac{d}{d\alpha} \left[\frac{d}{d\alpha} [F(\alpha)] \right]$$

$$= \frac{d}{d\alpha} \left[\int_{-\infty}^{\infty} (ix) e^{i\alpha x} f(x) dx \right]$$

$$= \int_{-\infty}^{\infty} f(ix) \frac{d}{d\alpha} (e^{i\alpha x}) f(x) dx$$

$$= \int_{-\infty}^{\infty} (ix)^2 e^{i\alpha x} f(x) dx$$

$$= (i)^2 \int_{-\infty}^{\infty} x^2 e^{i\alpha x} f(x) dx$$

$$\frac{d^2}{d\alpha^2} [F(\alpha)] = (i)^2 \int_{-\infty}^{\infty} e^{i\alpha x} [x^2 f(x)] dx$$

$$\text{Hence, } \boxed{\frac{d^2}{d\alpha^2} [F(\alpha)] = (i)^2 F[x^2 f(x)]}$$

$$\text{Ily } \frac{d^3}{d\alpha^3} [F(\alpha)] = (i)^3 F[x^3 f(x)]$$

In general

$$\frac{d^n}{d\alpha^n} [F(\alpha)] = (i)^n F[x^n f(x)]$$

$$\therefore F[x^n f(x)] = (-i)^n \frac{d^n}{d\alpha^n} [F(\alpha)]$$

1. Find the fourier transform of $f(x) = x$ if $|x| \leq a$
 $= 0$ if $|x| > a$

Sol:- Given $f(x) = x$ if $|x| \leq a$ i.e. $-a \leq x \leq a$ $\left[\begin{array}{l} |x| > a \\ x > a, x < -a \end{array} \right]$
 $= 0$ if $|x| > a$ i.e. $-\infty < x < -a$ and $a < x < \infty$

The fourier transform of $f(x)$ is given by

$$\begin{aligned} F[f(x)] &= \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx \\ &= \int_{-\infty}^{-a} f(x) e^{i\alpha x} dx + \int_{-a}^a f(x) e^{i\alpha x} dx + \int_a^{\infty} f(x) e^{i\alpha x} dx \\ &= \int_{-\infty}^{-a} 0 \cdot e^{i\alpha x} dx + \int_{-a}^a x e^{i\alpha x} dx + \int_a^{\infty} 0 \cdot e^{i\alpha x} dx \\ &= \int_{-a}^a x e^{i\alpha x} dx \quad \left[\int u v = u v_1 - u' v_2 + \dots \right] \\ &= \left[x \frac{e^{i\alpha x}}{i\alpha} - (1) \frac{e^{i\alpha x}}{(i\alpha)^2} \right]_{-a}^a \quad \left[\begin{array}{l} u = x \quad v = e^{i\alpha x} \\ u' = 1, v_1 = \frac{e^{i\alpha x}}{i\alpha}, v_2 = \frac{e^{i\alpha x}}{(i\alpha)^2} \end{array} \right] \\ &= \left[\left(\frac{a e^{i\alpha a}}{i\alpha} - \frac{e^{i\alpha a}}{i^2 \alpha^2} \right) - \left(-a \frac{e^{-i\alpha a}}{i\alpha} - \frac{e^{-i\alpha a}}{i^2 \alpha^2} \right) \right] \\ &= \frac{a}{i\alpha} e^{i\alpha a} - \frac{e^{i\alpha a}}{(-1)\alpha^2} + \frac{a e^{-i\alpha a}}{i\alpha} + \frac{e^{-i\alpha a}}{(-1)\alpha^2} \\ &= \frac{a}{i\alpha} [e^{i\alpha a} + e^{-i\alpha a}] + \frac{e^{i\alpha a}}{\alpha^2} - \frac{e^{-i\alpha a}}{\alpha^2} \\ &= \frac{a}{i\alpha} (2 \cos \alpha a) + \frac{1}{\alpha^2} [e^{i\alpha a} - e^{-i\alpha a}] \\ &= \frac{2a \cos \alpha a}{i\alpha} + \frac{2i \sin \alpha a}{\alpha^2} \quad \left[\begin{array}{l} \because e^{i\theta} + e^{-i\theta} = 2 \cos \theta \\ e^{i\theta} - e^{-i\theta} = 2i \sin \theta \end{array} \right] \\ &= \frac{2i \alpha \cos \alpha a}{i^2 \alpha^2} + \frac{2i \sin \alpha a}{\alpha^2} \\ &= -\frac{2\alpha \cos \alpha a}{\alpha^2} + \frac{2i \sin \alpha a}{\alpha^2} \end{aligned}$$

Hence $F[f(x)] = \frac{2i}{\alpha^2} [\sin \alpha a - \alpha \cos \alpha a]$

2. Find the Fourier transform of $f(x) = \frac{1}{2a}$ for $|x| \leq a$
 $= 0$ for $|x| > a$, $a > 0$.

Sol:- The F.T. of $f(x)$ is

$$\begin{aligned} F[f(x)] &= \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx \\ &= \int_{-\infty}^{-a} 0 \cdot e^{i\alpha x} dx + \int_{-a}^a \frac{1}{2a} e^{i\alpha x} dx + \int_a^{\infty} 0 \cdot e^{i\alpha x} dx \\ &= \frac{1}{2a} \int_{-a}^a e^{i\alpha x} dx \\ &= \frac{1}{2a} \left[\frac{e^{i\alpha x}}{i\alpha} \right]_{-a}^a \\ &= \frac{1}{2a} \left[\frac{e^{i\alpha a} - e^{-i\alpha a}}{i\alpha} \right] \quad \left[\because e^{i\theta} - e^{-i\theta} = 2i \sin \theta \right] \\ &= \frac{1}{2a} \left[\frac{2i \sin \alpha a}{i\alpha} \right] \end{aligned}$$

$$F[f(x)] = \frac{\sin \alpha a}{\alpha a}$$

3. Find the Fourier transform of $f(x) = \cos x$ for $|x| \leq a$
 $= 0$ for $|x| > a$, $a > 0$.

Sol:- The F.T. of $f(x)$ is $F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx$

$$\begin{aligned} \text{i.e. } F[f(x)] &= \int_{-\infty}^{-a} 0 \cdot e^{i\alpha x} dx + \int_{-a}^a \cos x e^{i\alpha x} dx + \int_a^{\infty} 0 \cdot e^{i\alpha x} dx \\ &= \int_{-a}^a e^{i\alpha x} \cos x dx \quad \left[\because \int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) \right] \\ &= \left[\frac{e^{i\alpha x}}{(i\alpha)^2 + 1} (i\alpha \cos x + \sin x) \right]_{-a}^a \\ &= \frac{e^{i\alpha a}}{-\alpha^2 + 1} (i\alpha \cos a + \sin a) - \frac{e^{-i\alpha a}}{-\alpha^2 + 1} (i\alpha \cos a - \sin a) \\ &= \frac{1}{1 - \alpha^2} i\alpha \cos a [e^{i\alpha a} - e^{-i\alpha a}] + \frac{\sin a}{1 - \alpha^2} [e^{i\alpha a} + e^{-i\alpha a}] \\ &= \frac{1}{1 - \alpha^2} i\alpha \cos a (2i \sin \alpha a) + \frac{\sin a}{1 - \alpha^2} (2 \cos \alpha a) \quad \left[\begin{aligned} e^{i\theta} - e^{-i\theta} &= 2i \sin \theta \\ e^{i\theta} + e^{-i\theta} &= 2 \cos \theta \end{aligned} \right] \\ &= \frac{-2\alpha \cos a \sin \alpha a}{1 - \alpha^2} + \frac{2 \sin a \cos \alpha a}{1 - \alpha^2} \end{aligned}$$

$$\therefore F[f(x)] = \frac{2}{1 - \alpha^2} [\sin a \cos \alpha a - \alpha \cos a \sin \alpha a]$$

4. Find the Fourier transform of $f(x) = 1-x^2$, if $|x| < 1$
 $= 0$, if $|x| > 1$

and use it to evaluate $\int_0^{\infty} \frac{x \cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx$.

Sol:- Fourier transform of $f(x)$ is given by $\left[\begin{array}{l} \because |x| < 1, -1 < x < 1 \\ |x| > 1, x > 1, x < -1 \end{array} \right]$

$$F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx$$

$$= \int_{-\infty}^{-1} 0 \cdot e^{i\alpha x} dx + \int_{-1}^1 (1-x^2) e^{i\alpha x} dx + \int_1^{\infty} 0 \cdot e^{i\alpha x} dx$$

$$= \int_{-1}^1 (1-x^2) e^{i\alpha x} dx$$

$$= \left[(1-x^2) \frac{e^{i\alpha x}}{i\alpha} - (-2x) \frac{e^{i\alpha x}}{(i\alpha)^2} + (-2) \frac{e^{i\alpha x}}{(i\alpha)^3} \right]_{-1}^1$$

$$\left[\begin{array}{l} u = 1-x^2, v = e^{i\alpha x} \\ u' = -2x, v_1 = \frac{e^{i\alpha x}}{i\alpha} \\ u'' = -2, v_2 = \frac{e^{i\alpha x}}{(i\alpha)^2} \\ \int uv = uv_1 - u'v_2 + u''v_3 \end{array} \right]$$

$$= \left[\left(0 \cdot \frac{e^{i\alpha}}{i\alpha} + 2(1) \frac{e^{i\alpha}}{(i\alpha)^2} - 2 \frac{e^{i\alpha}}{(i\alpha)^3} \right) - \left(0 \cdot \frac{e^{-i\alpha}}{i\alpha} + 2(-1) \frac{e^{-i\alpha}}{(i\alpha)^2} - 2 \frac{e^{-i\alpha}}{(i\alpha)^3} \right) \right]$$

$$= \frac{2 e^{i\alpha}}{i^2 \alpha^2} - \frac{2 e^{i\alpha}}{i^3 \alpha^3} + \frac{2 e^{-i\alpha}}{i^2 \alpha^2} + \frac{2 e^{-i\alpha}}{i^3 \alpha^3} \quad \left[\begin{array}{l} i^2 = -1 \\ i^3 = i^2 i = -i \end{array} \right]$$

$$= -\frac{2 e^{i\alpha}}{\alpha^2} - \frac{2 e^{i\alpha}}{(-i) \alpha^3} + \frac{2 e^{-i\alpha}}{(-1) \alpha^2} + \frac{2 e^{-i\alpha}}{(-i) \alpha^3}$$

$$= -\frac{2}{\alpha^2} [e^{i\alpha} + e^{-i\alpha}] + \frac{2}{i \alpha^3} [e^{i\alpha} - e^{-i\alpha}]$$

$$= -\frac{2}{\alpha^2} (2 \cos \alpha) + \frac{2}{i \alpha^3} (2i \sin \alpha)$$

$$\left[\begin{array}{l} e^{i\theta} + e^{-i\theta} = 2 \cos \theta \\ e^{i\theta} - e^{-i\theta} = 2i \sin \theta \end{array} \right]$$

$$= -\frac{4}{\alpha^2} \cos \alpha + \frac{4 \sin \alpha}{\alpha^3}$$

$$F[f(x)] = -\frac{4}{\alpha^3} [\alpha \cos \alpha - \sin \alpha] = F(\alpha) \quad \text{--- (1)}$$

By inversion formula for Fourier transform, $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha) e^{-i\alpha x} d\alpha$

$$\text{i.e. } f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} -\frac{4}{\alpha^3} [\alpha \cos \alpha - \sin \alpha] (\cos \alpha x - i \sin \alpha x) d\alpha$$

$$= -\frac{2}{\pi} \int_{-\infty}^{\infty} \frac{(\alpha \cos \alpha - \sin \alpha) \cos \alpha x}{\alpha^3} d\alpha + \frac{2i}{\pi} \int_{-\infty}^{\infty} \frac{(\alpha \cos \alpha - \sin \alpha) \sin \alpha x}{\alpha^3} d\alpha$$

$$= -\frac{4}{\pi} \int_0^{\infty} \frac{(\alpha \cos \alpha - \sin \alpha) \cos \alpha x}{\alpha^3} d\alpha$$

$\left[\begin{array}{l} \because \text{the integrand in the 1st} \\ \text{integral is even and that} \\ \text{in the second integral is} \\ \text{odd} \end{array} \right]$

$$\Rightarrow -\frac{4}{\pi} \int_0^{\infty} \left(\frac{\alpha \cos \alpha - \sin \alpha}{\alpha^3} \right) \cos \alpha x \, d\alpha = f(x)$$

$$\Rightarrow \int_0^{\infty} \left(\frac{\alpha \cos \alpha - \sin \alpha}{\alpha^3} \right) \cos \alpha x \, d\alpha = -\frac{\pi}{4} f(x) = -\frac{\pi}{4} (1-x^2), \text{ if } |x| < 1$$

$$= 0, \text{ if } |x| > 1$$

putting $x = \frac{1}{2}$, we have $\int_0^{\infty} \left(\frac{\alpha \cos \alpha - \sin \alpha}{\alpha^3} \right) \cos \left(\frac{\alpha}{2} \right) d\alpha = -\frac{\pi}{4} \left(1 - \frac{1}{4} \right)$

$$= -\frac{3\pi}{16}$$

Hence, $\boxed{\int_0^{\infty} \left(\frac{\alpha \cos \alpha - \sin \alpha}{\alpha^3} \right) \cos \left(\frac{\alpha}{2} \right) d\alpha = -\frac{3\pi}{16}}$ $\left[\int_a^b f(t) dt = \int_a^b f(x) dx \right]$

* Note:- If the function $f(x)$ is equal to $F(x)$ then $f(x)$ is called Self-reciprocal.

2. Show that the Fourier transform of $e^{-x^2/2}$ is self-reciprocal.

Let $f(x) = e^{-x^2/2}$ then

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{i\alpha x} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2} + i\alpha x} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} [x^2 - 2i\alpha x + (i\alpha)^2 - (i\alpha)^2]} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} [(x-i\alpha)^2 - i^2\alpha^2]} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} [(x-i\alpha)^2] + \frac{\alpha^2}{2}} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-i\alpha)^2}{2}} e^{\frac{\alpha^2}{2}} \, dx \quad \text{put } \frac{x-i\alpha}{\sqrt{2}} = t$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} e^{\frac{\alpha^2}{2}} \sqrt{2} \, dt$$

$$= \frac{e^{\frac{\alpha^2}{2}} \sqrt{2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} \, dt$$

$$F[f(x)] = \frac{\sqrt{2} e^{\frac{\alpha^2}{2}}}{\sqrt{2\pi}} \sqrt{\pi} = e^{\alpha^2/2} \text{ (or) } e^{-x^2/2}$$

$\therefore f(x)$ is self-reciprocal.

$$\left[\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t) e^{i\alpha t} \, dt \right] e^{-i\alpha x} \, d\alpha \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\alpha t} \, dt \right] e^{-i\alpha x} \, d\alpha \end{aligned} \right]$$

$$\left[\begin{aligned} \int_{-\infty}^{\infty} e^{-x^2} \, dx &= 2 \int_0^{\infty} e^{-x^2} \, dx \\ &= 2 \int_0^{\infty} e^{-t} \frac{1}{2} t^{-1/2} \, dt \quad x^2 = t \\ &= \int_0^{\infty} e^{-t} t^{-1/2} \, dt \quad x = \sqrt{t} \\ &= \Gamma\left(\frac{1}{2}\right) \\ &= \sqrt{\pi} \end{aligned} \right]$$

$$\left[\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi} \right]$$

6. Show that the Fourier transform of $f(x) = a - |x|$, for $|x| < a$
 $= 0$, for $|x| > a > 0$
 is $\sqrt{\frac{2}{\pi}} \left[\frac{1 - \cos ax}{x^2} \right]$. Hence deduce that $\int_0^\infty \left(\frac{\sin t}{t} \right)^2 dt = \frac{\pi}{2}$.

Sol:- The Fourier transform of $f(x)$ is given by

$$\begin{aligned}
 F[f(x)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx \\
 &= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-a} 0 \cdot e^{i\alpha x} dx + \int_{-a}^a (a - |x|) e^{i\alpha x} dx + \int_a^{\infty} 0 \cdot e^{i\alpha x} dx \right] \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-a}^a (a - |x|) e^{i\alpha x} dx \\
 &= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^a a e^{i\alpha x} dx - \int_{-a}^a |x| e^{i\alpha x} dx \right] \\
 &= \frac{1}{\sqrt{2\pi}} \left[a \left[\frac{e^{i\alpha x}}{i\alpha} \right]_{-a}^a - \left[\int_{-a}^0 -x e^{i\alpha x} dx + \int_0^a x e^{i\alpha x} dx \right] \right] \\
 &= \frac{1}{\sqrt{2\pi}} \left[\frac{a}{i\alpha} (e^{i\alpha a} - e^{-i\alpha a}) + \left[x \frac{e^{i\alpha x}}{i\alpha} - (1) \frac{e^{i\alpha x}}{(i\alpha)^2} \right]_{-a}^0 - \left[x \frac{e^{i\alpha x}}{i\alpha} - \frac{e^{i\alpha x}}{(i\alpha)^2} \right]_0^a \right] \\
 &= \frac{1}{\sqrt{2\pi}} \left[\frac{a}{i\alpha} e^{i\alpha a} - \frac{a}{i\alpha} e^{-i\alpha a} + \left[(0 - \frac{e^0}{-\alpha^2}) - \left(-\frac{a e^{-i\alpha a}}{i\alpha} - \frac{e^{-i\alpha a}}{-\alpha^2} \right) \right] \right. \\
 &\quad \left. - \left[\left(\frac{a e^{i\alpha a}}{i\alpha} - \frac{e^{i\alpha a}}{-\alpha^2} \right) - \left(0 - \frac{e^0}{-\alpha^2} \right) \right] \right] \\
 &= \frac{1}{\sqrt{2\pi}} \left[\frac{a}{i\alpha} e^{i\alpha a} - \frac{a}{i\alpha} e^{-i\alpha a} + \frac{1}{\alpha^2} + \frac{a e^{-i\alpha a}}{i\alpha} + \frac{e^{-i\alpha a}}{-\alpha^2} - \frac{a e^{i\alpha a}}{i\alpha} + \frac{e^{i\alpha a}}{-\alpha^2} + \frac{1}{\alpha^2} \right] \\
 &= \frac{1}{\sqrt{2\pi}} \left[\frac{2}{\alpha^2} - \frac{1}{\alpha^2} (e^{i\alpha a} + e^{-i\alpha a}) \right] \\
 &= \frac{1}{\sqrt{2\pi}} \left[\frac{2}{\alpha^2} - \frac{1}{\alpha^2} 2 \cos \alpha a \right] \quad [\because e^{i\theta} + e^{-i\theta} = 2 \cos \theta] \\
 &= \frac{2}{\sqrt{2\pi}} \left[\frac{1 - \cos \alpha a}{\alpha^2} \right] = \sqrt{\frac{2}{\pi}} \left[\frac{1 - \cos \alpha a}{\alpha^2} \right]
 \end{aligned}$$

$$\therefore F[f(x)] = \sqrt{\frac{2}{\pi}} \left[\frac{1 - \cos \alpha a}{\alpha^2} \right]$$

7. Find the Fourier transform of $e^{-a|x|}$ ($a > 0$) and hence show that

$$i) \int_0^{\infty} \frac{\cos px}{a^2 + p^2} dp = \frac{\pi}{2a} e^{-a|x|} \quad ii) F[x e^{-a|x|}] = \frac{i4ap}{(a^2 + p^2)^2}$$

Sol:- The Fourier transform of $f(x)$ is given by

$$F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{ipx} dx \quad [\because x = p]$$

Let $f(x) = e^{-a|x|}$ then

$$\begin{aligned} F[f(x)] &= \int_{-\infty}^{\infty} e^{-a|x|} e^{ipx} dx \\ &= \int_{-\infty}^{\infty} e^{-a|x|} (\cos px + i \sin px) dx \quad [\because e^{i\theta} = \cos \theta + i \sin \theta] \\ &= \int_{-\infty}^{\infty} e^{-a|x|} \cos px dx + i \int_{-\infty}^{\infty} e^{-a|x|} \sin px dx \\ &= 2 \int_0^{\infty} e^{-ax} \cos px dx + i(0) \quad \left[\begin{array}{l} \because \text{Integrand in 1st} \\ \text{integral is even and odd} \\ \text{in 2nd integral} \end{array} \right] \\ &= 2 \int_0^{\infty} e^{-ax} \cos px dx \quad [\because |x| = x \text{ in } (0, \infty)] \\ &= 2 \frac{a}{a^2 + p^2} = \frac{2a}{a^2 + p^2} = F(p) \quad \left[\because \int_0^{\infty} e^{-ax} \cos bx dx = \frac{a}{a^2 + b^2} \right] \end{aligned}$$

Deduction: i) By inverse Fourier transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(p) e^{-ipx} dp \quad [\because x = p]$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2a}{a^2 + p^2} e^{-ipx} dp \quad [\because \textcircled{1}]$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a}{a^2 + p^2} (\cos px - i \sin px) dp \quad [e^{-i\theta} = \cos \theta - i \sin \theta]$$

$$= \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{\cos px}{a^2 + p^2} dp - i \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{\sin px}{a^2 + p^2} dp$$

$$= \frac{2a}{\pi} \int_0^{\infty} \frac{\cos px}{a^2 + p^2} dp - i(0) \quad \left[\begin{array}{l} \because \text{Integrand in the 2nd} \\ \text{integral is odd} \end{array} \right]$$

$$\Rightarrow f(x) = \frac{2a}{\pi} \int_0^{\infty} \frac{\cos px}{a^2 + p^2} dp$$

$$\Rightarrow \frac{\pi}{2a} f(x) = \int_0^{\infty} \frac{\cos px}{a^2 + p^2} dp$$

$$\therefore \int_0^{\infty} \frac{\cos px}{a^2 + p^2} dp = \frac{\pi}{2a} e^{-a|x|}$$

ii) w.k.t. $F[x^n f(x)] = (-i)^n \frac{d^n}{dp} [F(p)] \quad [\because \alpha = p]$

for $n=1$, $F[x f(x)] = -i \frac{d}{dp} [F(p)]$

$\therefore F[x e^{-a|x|}] = -i \frac{d}{dp} \left[\frac{2a}{a^2 + p^2} \right] \quad [\because \textcircled{1}]$

$= -i(2a) \frac{d}{dp} \left[\frac{1}{a^2 + p^2} \right]$

$= -i(2a) \left[-\frac{1}{(a^2 + p^2)^2} (0 + 2p) \right]$

$= \frac{i 4 a p}{(a^2 + p^2)^2}$

8. Find the Fourier transform of e^{-x^r} . Hence find the Fourier transform of i) e^{-ax^r} , ($a > 0$) ii) $e^{-x^r/2}$ iii) $e^{-4(x-3)^r}$ iv) $e^{-x^r} \cos 2x$.

Sol:- The Fourier transform of $f(x) = e^{-x^r}$ is given by

$F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx$

$= \int_{-\infty}^{\infty} e^{-x^r} e^{i\alpha x} dx$

$= \int_{-\infty}^{\infty} e^{-x^r + i\alpha x} dx$

$= \int_{-\infty}^{\infty} e^{-(x^r - i\alpha x)} dx$

$= \int_{-\infty}^{\infty} e^{-(x^r - 2x(\frac{i\alpha}{2}) + (\frac{i\alpha}{2})^r - (\frac{i\alpha}{2})^r)} dx$

$= \int_{-\infty}^{\infty} e^{-(x - i\alpha/2)^r} + (\frac{i\alpha}{2})^r dx$

$= \int_{-\infty}^{\infty} e^{-(x - \frac{i\alpha}{2})^r} e^{\frac{i^r \alpha^r}{4}} dx \quad [i^r = -1]$

$= \int_{-\infty}^{\infty} e^{-t^r} e^{\frac{\alpha^r}{4}} dt$

$= e^{\alpha^r/4} \int_{-\infty}^{\infty} e^{-t^r} dt$

$= 2 e^{\alpha^r/4} \int_0^{\infty} e^{-t^r} dt$

$F[f(x)] = 2 e^{\alpha^r/4} \frac{\sqrt{\pi}}{2} = \sqrt{\pi} e^{\alpha^r/4}$

$\therefore F[f(x)] = \sqrt{\pi} e^{\alpha^r/4} = F(\alpha) - \textcircled{1}$

put $x - \frac{i\alpha}{2} = t$

$\Rightarrow dx = dt$

As $x \rightarrow -\infty$, $t \rightarrow -\infty$

As $x \rightarrow \infty$, $t \rightarrow \infty$

$\therefore \int_{-\infty}^{\infty} e^{-t^r} dt, \quad t^r = z$

$= \int_0^{\infty} e^{-z} \frac{1}{z^{\frac{1}{r}}} \frac{dz}{r} \quad t = \sqrt[r]{z}$

$= \frac{1}{r} \int_0^{\infty} e^{-z} z^{\frac{1}{r}-1} dz \quad dt = \frac{1}{r} z^{\frac{1}{r}-1} dz$

$= \frac{1}{r} \Gamma\left(\frac{1}{r}\right) = \frac{\sqrt{\pi}}{r}$

$$8. \text{ i) } e^{-ax^r} = e^{-(\sqrt{a}x)^r} = f(\sqrt{a}x)$$

By change of scale property, we have

$$F[f(ax)] = \frac{1}{a} F\left[\frac{x}{a}\right]$$

$$\Rightarrow F[e^{-ax^r}] = F[e^{-(\sqrt{a}x)^r}] \\ = \frac{1}{\sqrt{a}} \sqrt{\pi} e^{-\frac{1}{4}\left(\frac{x}{\sqrt{a}}\right)^r} = \sqrt{\frac{\pi}{a}} \cdot e^{-\frac{x^r}{4a}}$$

ii) Put $a = \frac{1}{2}$ in (i), we have

$$F[e^{-x^r/2}] = \sqrt{\frac{\pi}{\frac{1}{2}}} e^{-\frac{x^r}{4(\frac{1}{2})}} = \sqrt{2\pi} e^{-x^r/2}$$

$$\text{iii) } e^{-4x^r} = e^{-(2x)^r} = f(2x)$$

By change of scale property, we have

$$F[f(2x)] = \frac{1}{2} F\left[\frac{x}{2}\right]$$

$$= \frac{1}{2} \sqrt{\pi} e^{-\frac{1}{4}\left(\frac{x}{2}\right)^r} \\ = \frac{\sqrt{\pi}}{2} e^{-\frac{x^r}{16}} \quad \text{--- I}$$

By shifting property $F[f(x-a)] = e^{iax} F(x)$

$$\therefore F[e^{-4(x-3)^r}] = e^{i3x} \frac{\sqrt{\pi}}{2} e^{-x^r/16} \quad \left[\begin{array}{l} \text{Here } F(x) \text{ is the F.T.} \\ \text{of } f(2x) \end{array} \right] \\ = \frac{\sqrt{\pi}}{2} e^{3ix - \frac{x^r}{16}}$$

iv) By modulation theorem

$$F[f(x) \cos ax] = \frac{1}{2} [F(x+a) + F(x-a)]$$

$$F[e^{-x^r} \cos 2x] = \frac{1}{2} \left[\sqrt{\pi} e^{-\frac{1}{4}(x+2)^r} + \sqrt{\pi} e^{-\frac{1}{4}(x-2)^r} \right] \\ = \frac{\sqrt{\pi}}{2} \left[e^{-\frac{1}{4}(x+2)^r} + e^{-\frac{1}{4}(x-2)^r} \right]$$

H.W. 1) Find the Fourier transform of $f(x) = a^r - x^r, |x| < 1$
 $= 0, |x| > 1$

Deduce that $\int_0^\infty \frac{\sin t - t \cos t}{t^3} dt = \frac{\pi}{4}$

[Hint for deduction taking $a=1$ & $x=0$]

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9. Find the Fourier sine and cosine transform of $2e^{-5x} + 5e^{-2x}$.

Sol: let $f(x) = 2e^{-5x} + 5e^{-2x}$. $f(x)$ is given.

The Fourier sine transform of $f(x)$ is given by

$$f_s(\alpha) = \int_0^{\infty} f(x) \sin \alpha x \, dx$$

$$= \int_0^{\infty} (2e^{-5x} + 5e^{-2x}) \sin \alpha x \, dx$$

$$= 2 \int_0^{\infty} e^{-5x} \sin \alpha x \, dx + 5 \int_0^{\infty} e^{-2x} \sin \alpha x \, dx$$

$$= 2 \left[\frac{e^{-5x}}{\alpha^2 + 25} (-5 \sin \alpha x - \alpha \cos \alpha x) \right]_0^{\infty} + 5 \left[\frac{e^{-2x}}{\alpha^2 + 4} (-2 \sin \alpha x - \alpha \cos \alpha x) \right]_0^{\infty}$$

$$= \frac{2}{\alpha^2 + 25} \left[0 - e^{-0} (-5(0) - \alpha(1)) \right] + \frac{5}{\alpha^2 + 4} \left[0 - e^{-0} (2(0) - \alpha(1)) \right]$$

$$= \frac{2\alpha}{\alpha^2 + 25} + \frac{5\alpha}{\alpha^2 + 4} \quad \left[\because \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) \right]$$

To find Fourier cosine transform,
the Fourier cosine transform of $f(x)$ is

$$\begin{aligned}
 F_c(\alpha) &= \int_0^{\infty} f(x) \cos \alpha x \, dx \\
 &= \int_0^{\infty} (2e^{-5x} + 5e^{-2x}) \cos \alpha x \, dx \\
 &= 2 \int_0^{\infty} e^{-5x} \cos \alpha x \, dx + 5 \int_0^{\infty} e^{-2x} \cos \alpha x \, dx \\
 &= 2 \left[\frac{e^{-5x}}{\alpha^2 + 5^2} (-5 \cos \alpha x + \alpha \sin \alpha x) \right]_0^{\infty} + 5 \left[\frac{e^{-2x}}{\alpha^2 + 4} (-2 \cos \alpha x + \alpha \sin \alpha x) \right]_0^{\infty} \\
 &= 2 \left[0 - \frac{1}{\alpha^2 + 25} (-5(1) + 0) \right] + \frac{5}{\alpha^2 + 4} \left(0 - 1(-2(1) + 0) \right) \\
 &= \frac{10}{\alpha^2 + 25} + \frac{10}{\alpha^2 + 4}
 \end{aligned}$$

Ex. 10. Find the Fourier sine transform of e^{-1x} and

hence ST $\int_0^{\infty} \frac{x \sin mx}{1+x^2} \, dx = \frac{\pi}{2} e^{-m}, m > 0$

Sol:- let $f(x) = e^{-1x}$

Fourier sine transform of $f(x)$ is

$$\begin{aligned}
 F_s(\alpha) &= \int_0^{\infty} f(x) \sin \alpha x \, dx \\
 &= \int_0^{\infty} e^{-x} \sin \alpha x \, dx
 \end{aligned}$$

$$= \frac{e^{-x}}{1^2 + \alpha^2} \left[-\sin \alpha x - \alpha \cos \alpha x \right]_0^{\infty}$$

$$= \frac{1}{\alpha^2 + 1} \left[0 - e^0 (-\alpha(1)) \right] = \frac{\alpha}{\alpha^2 + 1}$$

$$F_s(\alpha) = \frac{\alpha}{\alpha^2 + 1}$$

FX [Deduction:-] By the inverse Fourier sine transform of $f_s(x)$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} f_s(\alpha) \sin \alpha x \, d\alpha$$

$$\therefore e^{-|x|} = \frac{2}{\pi} \int_0^{\infty} \frac{\alpha}{\alpha^2 + 1} \sin \alpha x \, d\alpha$$

changing x to m

$$e^{-|m|} = \frac{2}{\pi} \int_0^{\infty} \frac{\alpha}{\alpha^2 + 1} \sin \alpha m \, d\alpha$$

$$\int_0^{\infty} \frac{\alpha}{\alpha^2 + 1} \sin \alpha m \, d\alpha = \frac{\pi}{2} e^{-|m|}$$

$$(or) \int_0^{\infty} \frac{x}{x^2 + 1} \sin mx \, dx = \frac{\pi}{2} e^{-m}, \quad m > 0$$

311. Find the Fourier cosine transform of $f(x) = x$

* for $0 < x < 1$

$= 2-x$ for $1 < x < 2$

$= 0$ for $x > 2$

Sol:- The Fourier cosine transform of $f(x)$ is given by

$$F_c(\alpha) = \int_0^{\infty} f(x) \cos \alpha x \, dx$$

$$= \int_0^1 x \cos \alpha x \, dx + \int_1^2 (2-x) \cos \alpha x \, dx + \int_2^{\infty} 0 \, dx$$

$$= \left[x \cdot \frac{\sin \alpha x}{\alpha} + \frac{\cos \alpha x}{\alpha^2} \right]_0^1 + \left[(2-x) \frac{\sin \alpha x}{\alpha} + \frac{(-1) \cos \alpha x}{\alpha^2} \right]_1^2$$

$$= \left[\frac{1}{\alpha} \sin \alpha + \frac{\cos \alpha}{\alpha^2} - \left(0 + \frac{1}{\alpha^2} \right) \right] + \left[0 - \frac{1}{\alpha^2} \cos 2\alpha - \left(\frac{\sin \alpha}{\alpha} - \frac{\cos \alpha}{\alpha^2} \right) \right]$$

$$= \frac{\sin \alpha}{\alpha} + \frac{\cos \alpha}{\alpha^2} - \frac{1}{\alpha^2} - \frac{\cos 2\alpha}{\alpha^2} - \frac{\sin \alpha}{\alpha} + \frac{\cos \alpha}{\alpha^2}$$

$$= \frac{2 \cos \alpha}{\alpha^2} - \frac{1}{\alpha^2} - \frac{\cos 2\alpha}{\alpha^2} = \frac{1}{\alpha^2} [2 \cos \alpha - \cos 2\alpha - 1]$$

$$= \frac{1}{\alpha^2} [2 \cos \alpha - (2 \cos^2 \alpha - 1) - 1]$$

$$= \frac{2\cos x - 2\cos^3 x}{x^2} : \frac{2\cos x (1 - \cos^2 x)}{x^2}$$

Q12. Find the Fourier sine transform of $\frac{e^{-ax}}{x}$

Sol: let $f(x) = \frac{e^{-ax}}{x}$

$$F_s(x) = \int_0^{\infty} f(x) \sin ax \, dx$$

$$= \int_0^{\infty} \frac{e^{-ax}}{x} \sin ax \, dx \quad \text{--- (1)}$$

diff (1) w.r.t a

$$\frac{d}{da} [F_s(x)] = \int_0^{\infty} \frac{e^{-ax}}{x} \cos ax \cdot x \, dx$$

$$= \int_0^{\infty} e^{-ax} \cos ax \, dx$$

$$\frac{d}{da} [F_s(x)] = \frac{a}{a^2 + x^2}$$

$$\left[\int_0^{\infty} e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2} \right]$$

Integrating on b.side

$$F_s(x) = \int \frac{a}{a^2 + x^2} da$$

$$= \frac{a}{x} \cdot \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$$

$$F_s(x) = \tan^{-1}\left(\frac{x}{a}\right) + c$$

if $x=0$ then $F_s(x)=0$

$$0 = \tan^{-1}(0) + c$$

$$0 = 0 + c \Rightarrow c=0$$

$$\therefore f_s(x) = \tan^{-1}\left(\frac{x}{a}\right)$$

fourier sine and cosine transform of

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Q13. Find the

x^{n-1} $n > 0$

Sol:

let $f(x) = x^{n-1}$

then $F_s[f(x)] = \int_0^{\infty} f(x) \sin ax \, dx$

then $F_s[f(x)] = \int_0^{\infty} f(x) \sin ax \, dx$

$$F_S [x^{n-1}] = \int_0^{\infty} x^{n-1} \sin \alpha x \, dx \quad \text{--- (1)}$$

$$F_C [x^{n-1}] = \int_0^{\infty} x^{n-1} \cos \alpha x \, dx$$

$$F_C [x^{n-1}] = \int_0^{\infty} x^{n-1} \cos \alpha x \, dx \quad \text{--- (2)}$$

(2) + (1)

we get $F_C [x^{n-1}] + i F_S [x^{n-1}] = \int_0^{\infty} x^{n-1} \cos \alpha x \, dx + i \int_0^{\infty} x^{n-1} \sin \alpha x \, dx$

$$= \int_0^{\infty} x^{n-1} [\cos \alpha x + i \sin \alpha x] \, dx$$

$$= \int_0^{\infty} x^{n-1} e^{i \alpha x} \, dx$$

$$= \int_0^{\infty} e^{i \alpha x} x^{n-1} \, dx$$

$$= \int_0^{\infty} e^{-t} \left(\frac{i}{\alpha}\right)^n \left(\frac{i}{\alpha}\right) \, dt$$

$$= \left(\frac{i}{\alpha}\right)^n \int_0^{\infty} e^{-t} t^{n-1} \, dt$$

$$= \left(\frac{i}{\alpha}\right)^n \Gamma(n)$$

$$= \frac{1}{\alpha^n} \left[\cos \frac{n\pi}{2} + i \sin \frac{n\pi}{2} \right] \Gamma(n)$$

$$\left[\begin{array}{l} \text{By De Moivre's theorem} \\ (e^{i\theta})^n = \cos n\theta + i \sin n\theta \end{array} \right]$$

$$F_C [x^{n-1}] + i F_S [x^{n-1}] = \frac{\Gamma(n)}{\alpha^n} \left[\cos \frac{n\pi}{2} + i \sin \frac{n\pi}{2} \right]$$

comparing the real and imaginary parts, we get

$$F_C [x^{n-1}] = \frac{\Gamma(n)}{\alpha^n} \cos \frac{n\pi}{2}$$

$$F_S [x^{n-1}] = \frac{\Gamma(n)}{\alpha^n} \sin \frac{n\pi}{2}$$

Q14. Find the Fourier cosine transform of e^{-x^2} then

Sol: let $f(x) = e^{-x^2}$

$$F_C(f(x)) = \int_0^{\infty} f(x) \cos \alpha x \, dx$$

$$= \int_0^{\infty} e^{-x^2} \cos \alpha x \, dx$$

let $f(x) = e^{-x^2}$

$F_c[f(x)] = \int_0^\infty f(x) \cos \alpha x \, dx$

$= \int_0^\infty e^{-x^2} \cos \alpha x \, dx = \frac{I}{2}$

Diff w.r.t α we get

$\frac{dI}{d\alpha} = \int_0^\infty e^{-x^2} [-x \sin \alpha x] \, dx$

$= \frac{1}{2} \int_0^\infty (-2x) e^{-x^2} \sin \alpha x \, dx$

$= \frac{1}{2} \int_0^\infty d[e^{-x^2}] \sin \alpha x \, dx$

$$= \frac{1}{2} \left[\sin \alpha x e^{-x^2} \right]_0^{\infty} - \int_0^{\infty} \alpha \cos \alpha x e^{-x^2} dx$$

$$= \frac{1}{2} (0-0) - \frac{\alpha}{2} \int_0^{\infty} e^{-x^2} \cos \alpha x dx \quad \text{--- form ①}$$

$$\frac{dI}{d\alpha} = -\frac{\alpha}{2} I$$

$$\frac{dI}{I} = -\frac{\alpha}{2} d\alpha$$

$$\log I = -\frac{\alpha^2}{4} + \log c$$

$$\log \frac{I}{c} = -\frac{\alpha^2}{4}$$

$$\frac{I}{c} = e^{-\alpha^2/4}$$

$$I = c e^{-\alpha^2/4} \quad \text{--- ②}$$

$$\text{If } \alpha=0 \text{ then } I = \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$\frac{\sqrt{\pi}}{2} = c \cdot e^{-0} \Rightarrow c = \frac{\sqrt{\pi}}{2}$$

$$\text{Sub } c = \frac{\sqrt{\pi}}{2} \text{ in ②}$$

$$I = f_c(f(x)) = \frac{\sqrt{\pi}}{2} e^{-\alpha^2/4}$$

Ex. 15. Find the Fourier cosine transform of

$$f(x) = \frac{1}{1+x^2}$$

Sol:

$$\text{let } f(x) = \frac{1}{1+x^2}$$

$$f_c(f(x)) = \int_0^{\infty} f(x) \cos \alpha x dx$$

$$= \int_0^{\infty} \frac{1}{1+x^2} \cos \alpha x dx \quad \text{--- ①}$$

Diff ① w.r.t α

$$\frac{dI}{d\alpha} = \int_0^{\infty} \frac{-x \sin \alpha x}{1+x^2} dx$$

$$= - \int_0^{\infty} \frac{x^2 \sin \alpha x}{x(1+x^2)} dx$$

$$= - \int_0^{\infty} \frac{(1+x^2-1) \sin \alpha x}{x(1+x^2)} dx$$

$$= - \int_0^{\infty} \frac{\sin \alpha x}{x} dx + \int_0^{\infty} \frac{\sin \alpha x}{x(1+x^2)} dx$$

$$= -\frac{\pi}{2} + \int_0^{\infty} \frac{\sin \alpha x}{x(1+x^2)} dx \quad \text{--- (2)}$$

Diff (2) w.r.t x then

$$\frac{d^2 I}{dx^2} = 0 + \int_0^{\infty} \frac{x \cos \alpha x}{x(1+x^2)} dx$$

$$\frac{d^2 I}{dx^2} = 0$$

$$D^2 I = 0$$

$$(D^2 - 1) I = 0 \quad \text{--- (3)}$$

The General Solution of (3) is

$$I = c_1 e^{-x} + c_2 e^x \quad \text{--- (4)}$$

If $x=0$ then $I = \int_0^{\infty} \frac{1}{1+x^2} dx$

$$= (\tan^{-1} x)_0^{\infty} = \frac{\pi}{2}$$

$$c_1 e^{\frac{\pi}{2}} = c_1 + c_2 \quad \text{--- (5)}$$

Diff (4) w.r.t x we get

$$\frac{dI}{dx} = -c_1 e^{-x} + c_2 e^x$$

If $x=0$ then $\frac{dI}{dx} = 0$

$$0 = -c_1 + c_2 \quad \text{--- (6)}$$

Solving (5) & (6)

$$2c_2 = \frac{\pi}{2} \Rightarrow c_2 = \frac{\pi}{4}$$

$$\Rightarrow c_1 = \frac{\pi}{4}$$

$$I = \frac{\pi}{4} e^{-x} + \frac{\pi}{4} e^x$$

$$= \frac{\pi}{4} (e^x + e^{-x}) = \frac{\pi}{2} \cosh x$$

16/7/14 Finite Fourier sine transforms

The finite let $f(x)$ be a function defined in $0 < x < l$ then the finite Fourier sine transform of $f(x)$ is defined by

$$F_s(n) = \int_0^l f(x) \sin \frac{n\pi x}{l} dx \quad \text{and the function}$$

$$f(x) = \frac{2}{l} \sum_{n=1}^{\infty} F_s(n) \sin \frac{n\pi x}{l} \quad \text{is called the}$$

inverse finite Fourier sine transform of $F_s(n)$

where n is an integer.

Finite Fourier cosine transform:- defined in $0 < x < l$, let $f(x)$ be a function of $f(x)$ is

then the Fourier cosine transform

$$\text{defined by } F_c(n) = \int_0^l f(x) \cos \frac{n\pi x}{l} dx$$

and the function $f(x)$ is given by

$$f(x) = \frac{1}{l} F_c(0) + \frac{2}{l} \sum_{n=1}^{\infty} F_c(n) \cos \frac{n\pi x}{l}$$

is called inverse finite Fourier cosine transform of $F_c(n)$

1. Find the finite Fourier sine and cosine transform

$$f(x) = x \quad \text{where } 0 < x < 4$$

Sol:- Given $f(x) = x$

To find finite Fourier sine transform of $f(x)$ is

for the finite Fourier sine transform given by

$$F_s(n) = \int_0^4 f(x) \sin \frac{n\pi x}{l} dx \quad [\because l = 4]$$

$$= \int_0^4 x \sin \frac{n\pi x}{4} dx$$

$$\left[\begin{array}{l} \text{Sub} \\ = uv - u'v_2 + \\ u = x, v = \sin \frac{n\pi x}{4} \end{array} \right]$$

$$= \left[x \cdot \cos \frac{n\pi x}{4} + \frac{\sin \frac{n\pi x}{4}}{(\frac{n\pi}{4})^2} \right]_0^4$$

$$= \left[-4 \cdot \frac{4}{n\pi} \cos n\pi - 0 + \left(\frac{4}{n\pi} \right)^2 \sin n\pi - 0 \right]$$

$$= -\frac{16}{n\pi} (-1)^n$$

$$F_s(n) = \frac{16}{n\pi} (-1)^{n+1}$$

To find finite Fourier cosine transform of $f(x)$ in $(0,1)$ is

The finite cosine transform given by $F_c(n) = \int_0^1 f(x) \cos n\pi x \, dx$

$$= \int_0^1 x \cos \frac{n\pi x}{4} \, dx$$

$$\begin{cases} \int u v = u v_1 - u_1 v_2 + \dots \\ u = x, v = \cos \frac{n\pi x}{4} \end{cases}$$

$$= \left[x \cdot \frac{\sin \frac{n\pi x}{4}}{\frac{n\pi}{4}} + (1) \frac{\cos \frac{n\pi x}{4}}{\left(\frac{n\pi}{4}\right)^2} \right]_0^1$$

$$= \left[4 \cdot \frac{4}{n\pi} \sin n\pi - 0 \right] + \left(\frac{4}{n\pi} \right)^2 [\cos n\pi - \cos 0]$$

$$= 0 + \left(\frac{4}{n\pi} \right)^2 [(-1)^n - 1]$$

$$F_c(n) = \frac{16}{n^2\pi^2} [(-1)^n - 1] \quad n > 0$$

If $n=0$ then $F_c(0) = \int_0^1 x \, dx$

$$= \left(\frac{x^2}{2} \right)_0^1 = \frac{1}{2}$$

$$\therefore F_c(n) = \frac{16}{n^2\pi^2} [(-1)^n - 1] \quad \text{if } n > 0 \quad n \neq 0$$

$$= \frac{1}{2} \quad \text{if } n=0$$

2. Find the finite Fourier cosine transform of

$$\left(1 - \frac{x}{\pi}\right)^2 \quad \text{in } (0, \pi).$$

Sol: Given $f(x) = \left(1 - \frac{x}{\pi}\right)^2$

The finite Fourier cosine transform

$$F_c(n) = \int_0^\pi f(x) \cos nx \, dx$$

$$= \int_0^\pi \left(1 - \frac{x}{\pi}\right)^2 \cos nx \, dx$$

$$= \left[\left(1 - \frac{x}{\pi}\right)^2 \frac{\sin nx}{n} - 2 \left(1 - \frac{x}{\pi}\right) \left(-\frac{1}{\pi}\right) \frac{-\cos nx}{n^2} + 2 \left(-\frac{1}{\pi}\right) \left(-\frac{1}{\pi}\right) \frac{-\sin nx}{n^3} \right]_0^\pi$$

$$= \left[\left(1 - \frac{x}{\pi}\right)^2 \frac{\sin nx}{n} - \frac{2}{n^2 \pi} \left(1 - \frac{x}{\pi}\right) \cos nx - \frac{2}{n^3 \pi^2} \sin nx \right]_0^\pi$$

$$= \left[0 - 0 - \frac{2}{n^3 \pi^2} \sin n\pi - \left[\frac{\sin n\pi}{n} - \frac{2}{n^2 \pi} \cos 0 - \frac{2}{n^3 \pi^2} \sin 0 \right] \right]$$

$$F_c(n) = \frac{2}{n^2 \pi} \quad \text{if } n > 0$$

$$\text{If } n=0 \quad F_c(0) = \int_0^\pi f(x) \cos dx$$

$$= \int_0^\pi \left(1 - \frac{x}{\pi}\right)^2 dx = \int_0^\pi \left(1 - \frac{x}{\pi}\right)^2 \left(-\frac{1}{\pi}\right) dx$$

$$= -\frac{1}{\pi} \left[\frac{\left(1 - \frac{x}{\pi}\right)^3}{3} \right]_0^\pi = \left[\frac{(f(x))^n f(x)}{n+1} \right]_0^\pi$$

$$= -\frac{\pi}{3} \cdot (0 - 1) = \frac{\pi}{3}$$

$$F_c(n) = \frac{2}{n^2 \pi} \quad \text{if } n > 0$$

$$= \frac{\pi}{3} \quad \text{if } n=0$$

3. Show that finite Fourier sine transform of $\frac{x}{\pi}$ is $\frac{(-1)^{n+1}}{n}$ in the interval $(0, \pi)$

The finite Fourier sine transform of $f(x)$ in $(0, \pi)$ is

$$F_s(n) = \int_0^\pi f(x) \sin nx \, dx$$

$$= \int_0^\pi \frac{x}{\pi} \sin nx \, dx$$

$$= \frac{1}{\pi} \left[x \frac{-\cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^\pi$$

$$= \frac{1}{\pi} \left[-\pi \frac{\cos n\pi}{n} - 0 + \frac{n\pi \sin n\pi}{n^2} - 0 \right]$$

$$= -\frac{(-1)^n}{n} = \frac{(-1)^{n+1}}{n} = F_s(n)$$

4. Find $f(x)$ if its finite Fourier cosine transform is $f_c(n) = \frac{\sin \frac{n\pi}{2}}{2n}$ $n = 1, 2, 3, \dots$
 $= \frac{\pi}{4}$ if $n=0$ and given $0 < x < 2\pi$

Sol:- From inverse finite Fourier cosine transform of

$$f_c(n) \text{ is given by}$$

$$f(x) = \frac{1}{2} f_c(0) + \sum_{n=1}^{\infty} f_c(n) \cos \frac{n\pi x}{L}$$

$$= \frac{1}{2} \left(\frac{\pi}{4} \right) + \sum_{n=1}^{\infty} \left(\frac{\sin \frac{n\pi}{2}}{2n} \right) \cos \frac{n\pi x}{2} \quad \left[\because L=2\pi \right]$$

$$= \frac{1}{8} + \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{n} \cos \frac{n\pi x}{2}$$

5 Find $f(x)$ if $f_s(n) = \frac{16(-1)^{n-1}}{n^3}$ if $0 < x < 8$
 when $1, 2, 3, \dots$

Sol:- Given $f_s(n) = \frac{16(-1)^{n-1}}{n^3}$ if $0 < x < 8$

and $L=8$

The finite inverse finite sine transform of $f_s(n)$ is given by

$$f(x) = \sum_{n=1}^{\infty} f_s(n) \sin \frac{n\pi x}{L}$$

$$= \sum_{n=1}^{\infty} \frac{16(-1)^{n-1}}{n^3} \sin \frac{n\pi x}{8}$$

$$= 4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \sin \frac{n\pi x}{8}$$

6 solve the integral equation $\int_0^{\infty} f(x) \cos dx = e^{-x}$
 and find $f(x)$.

Sol:- Given $f_c(x) = e^{-x}$
 By inverse Fourier cosine transform

$$f(x) = \frac{2}{\pi} \int_0^{\infty} f_c(x) \cos dx$$

$$= \frac{2}{\pi} \int_0^{\infty} e^{-x} \cos dx$$

$$= \frac{2}{\pi} \left[\frac{1}{1+x^2} \right] \because e^x \cos bx = \frac{x}{1+x^2}$$

Parseval's identity for fourier transform :-
 If $F(\alpha)$ & $G(\alpha)$ are fourier transforms of $f(x)$ and $g(x)$ respectively, then

$$(i) \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha) \overline{G(\alpha)} d\alpha = \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx \quad \text{where bar stands for complex conjugate}$$

$$(ii) \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\alpha)|^2 d\alpha = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

Parseval's identities for fourier sine transform
 If $F_s(\alpha)$ and $G_s(\alpha)$ are the fourier sine transforms of $f(x)$ and $g(x)$ respectively, then

$$(i) \frac{2}{\pi} \int_0^{\infty} F_s(\alpha) G_s(\alpha) d\alpha = \int_0^{\infty} f(x) g(x) dx$$

$$(ii) \frac{2}{\pi} \int_0^{\infty} |F_s(\alpha)|^2 d\alpha = \int_0^{\infty} |f(x)|^2 dx$$

Parseval's identity for fourier cosine transform
 If $F_c(\alpha)$ and $G_c(\alpha)$ are the fourier cosine transforms of $f(x)$ and $g(x)$ respectively, then

$$(i) \frac{2}{\pi} \int_0^{\infty} F_c(\alpha) G_c(\alpha) d\alpha = \int_0^{\infty} f(x) g(x) dx$$

$$(ii) \frac{2}{\pi} \int_0^{\infty} |F_c(\alpha)|^2 d\alpha = \int_0^{\infty} |f(x)|^2 dx$$

1) If $f(x) = 1, |x| < 1$ and $f(s) = \frac{2}{s} \sin s$ using
 $= 0, |x| > 1$
 Parseval's identity, put $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$.

Sol:- Given $F(s) = \frac{2}{s} \sin s$

Using Parseval's identity for Fourier transform

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$\text{i.e. } \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|\frac{2}{s} \sin s\right|^2 ds = \int_{-\infty}^{-1} 0 dx + \int_{-1}^1 1 dx + \int_1^{\infty} 0 dx$$

$$\Rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4 \sin^2 s}{s^2} ds = [x]_{-1}^1$$

$$\Rightarrow \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 s}{s^2} ds = [1 - (-1)] = 2$$

$$\Rightarrow \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 s}{s^2} ds = 2$$

$$\Rightarrow \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 s}{s^2} ds = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin^2 s}{s^2} ds = \pi \quad \left[\frac{\sin^2 s}{s^2} \text{ is an even function of } s \right]$$

$$\Rightarrow 2 \int_0^{\infty} \frac{\sin^2 s}{s^2} ds = \pi$$

$$\Rightarrow \int_0^{\infty} \frac{\sin^2 s}{s^2} ds = \frac{\pi}{2} \quad (\text{or}) \quad \int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}.$$

2). Show that the Fourier transform of $f(x) = 1 - x^2$, if $|x| < 1$
 $= 0$, if $|x| > 1$

is $\frac{4}{p^3} (\sin p - pc \cdot p)$. Using Parseval's identity prove that

$$\int_0^{\infty} \left(\frac{\sin x - x \cos x}{x^3}\right)^2 dx = \frac{\pi}{15}.$$

Sol:- For Fourier transform of $f(x)$, Refer 4th problem in (F.T.Y)

$$\text{i.e. } f(p) = \frac{4}{p^3} (\sin p - pc \cdot p) \quad \text{--- (1)}$$

Using Parseval's identity

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |F(p)|^2 dp = \int_{-\infty}^{\infty} |f(x)|^2 dx.$$

$$\text{i.e. } \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{4(\sin p - p \cos p)}{p^3} \right]^2 dp = \int_{-\infty}^1 0 dx + \int_{-1}^1 (1-x^2)^2 dx + \int_1^{\infty} 0 dx.$$

$$\text{i.e. } \frac{8}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin p - p \cos p}{p^3} \right)^2 dp = \int_{-1}^1 (1+x^4-2x^2) dx$$

$$\text{i.e. } \frac{8}{\pi} \cdot 2 \int_0^{\infty} \left(\frac{\sin p - p \cos p}{p^3} \right)^2 dp = \left[x + \frac{x^5}{5} - 2 \frac{x^3}{3} \right]_{-1}^1$$

$$\text{i.e. } \frac{16}{\pi} \int_0^{\infty} \left(\frac{\sin p - p \cos p}{p^3} \right)^2 dp = \left[\left(1 + \frac{1}{5} - \frac{2}{3} \right) - \left(-1 - \frac{1}{5} + \frac{2}{3} \right) \right]$$

$$\text{i.e. } \frac{16}{\pi} \int_0^{\infty} \left(\frac{\sin p - p \cos p}{p^3} \right)^2 dp = 2 + \frac{2}{5} - \frac{4}{3} = \frac{30+6-20}{15} = \frac{16}{15}$$

$$\text{i.e. } \frac{16}{\pi} \int_0^{\infty} \left(\frac{\sin p - p \cos p}{p^3} \right)^2 dp = \frac{16}{15}$$

$$\Rightarrow \frac{1}{\pi} \int_0^{\infty} \left(\frac{\sin p - p \cos p}{p^3} \right)^2 dp = \frac{1}{15}$$

$$\Rightarrow \int_0^{\infty} \left(\frac{\sin p - p \cos p}{p^3} \right)^2 dp = \frac{\pi}{15}$$

$$\text{(or)} \boxed{\int_0^{\infty} \left(\frac{\sin x - x \cos x}{x^3} \right)^2 dx = \frac{\pi}{15}}.$$

1) Find the F.T. of $f(x)$ defined by $f(x) = 1, |x| < a$
 $= 0, |x| > a$

$$\text{and hence p.t. } \int_0^{\infty} \frac{\sin^2 ax}{x^2} dx = \frac{\pi a}{2}.$$

2) Find the F.T. of $f(x) = 1 - |x|$ when $|x| < 1$
 $= 0$ when $|x| > 1$

$$\text{and hence deduce that } \int_0^{\infty} \left(\frac{\sin x}{x} \right)^4 dx = \frac{\pi}{3}.$$

2. Using Parseval's identity, Evaluate $\int_0^{\infty} \frac{x^2}{(a^2+x^2)^2} dx$

Sol:-

$$\int_0^{\infty} \frac{x^2}{(a^2+x^2)^2} dx = \int_0^{\infty} \frac{x}{a^2+x^2} \cdot \frac{x}{a^2+x^2} dx$$

let $f(x) = e^{-ax}$ then $F_S(\alpha) = \int_0^{\infty} e^{-ax} \sin \alpha x dx$

$$\frac{\alpha}{a^2 + \alpha^2}$$

By Parseval's identity for Fourier sine transform

$$\frac{2}{\pi} \int_0^{\infty} [F_S(\alpha)]^2 d\alpha = \int_0^{\infty} |f(x)|^2 dx$$

$$\frac{2}{\pi} \int_0^{\infty} \left(\frac{\alpha}{a^2 + \alpha^2} \right)^2 d\alpha = \int_0^{\infty} (e^{-ax})^2 dx$$

$$= \int_0^{\infty} e^{-2ax} dx$$

$$= \left(\frac{e^{-2ax}}{-2a} \right)_0^{\infty} = \frac{-1}{2a} (0 - e^0)$$

$$= \frac{1}{2a}$$

$$\Rightarrow \int_0^{\infty} \left(\frac{x^2}{a^2+x^2} \right)^2 dx = \frac{\pi}{4a}$$

$$(or) \int_0^{\infty} \frac{x^2}{(a^2+x^2)^2} dx = \frac{\pi}{4a}$$

3. using Parseval's identity, Evaluate $\int_0^{\infty} \frac{x^2}{(x^2+a^2)^2} dx$

Sol: $\int_0^{\infty} \frac{dx}{(a^2+x^2)^2} = \int_0^{\infty} \frac{a}{a^2+x^2} \cdot \frac{a}{a^2+x^2} dx$

let $f(x) = e^{-ax}$ then $F_c(\omega) = \int_0^{\infty} e^{-ax} \cos \omega x dx = \frac{a}{a^2+\omega^2}$

By Parseval's identity for Fourier transform

$$\frac{2}{\pi} \int_0^{\infty} |F_c(\omega)|^2 d\omega = \int_0^{\infty} |f(x)|^2 dx$$

$$= \frac{2}{\pi} \int_0^{\infty} \left(\frac{a}{a^2+\omega^2} \right)^2 d\omega = \int_0^{\infty} (e^{-ax})^2 dx$$

$$= \int_0^{\infty} e^{-2ax} dx$$

$$= \left(\frac{e^{-2ax}}{-2a} \right)_0^{\infty}$$

$$= \frac{1}{2a} (0 - e^0) = \frac{1}{2a}$$

$$= \int_0^{\infty} \frac{a^2}{(a^2+x^2)^2} dx = \frac{\pi}{4a}$$

$$\int_0^{\infty} \frac{dx}{(a^2+x^2)^2} = \frac{\pi}{4a^3}$$

$$\int_0^{\infty} \frac{dx}{(a^2+x^2)^2} = \frac{\pi}{4a^3} //$$

4. Using Parseval's identity, show that $\int_0^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx = \frac{\pi}{2(a+b)}$

Sol: $\int_0^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx = \int_0^{\infty} \frac{x}{x^2+a^2} \cdot \frac{x}{x^2+b^2} dx$

let $f(x) = e^{-ax}$ then $F_c(\omega) = \int_0^{\infty} e^{-ax} \cos \omega x dx = \frac{a}{a^2+\omega^2}$
 $g(x) = e^{-bx}$ then $G_c(\omega) = \frac{b}{b^2+\omega^2}$

By Parseval's identity for Fourier sine transform

$$\frac{2}{\pi} \int_0^{\infty} f_s(x) g_s(x) dx = \int_0^{\infty} f(x) g(x) dx$$

$$\frac{2}{\pi} \int_0^{\infty} \frac{x}{x^2+a^2} \frac{dx}{x^2+b^2} = \int_0^{\infty} e^{-ax} e^{-bx} dx$$

$$\Rightarrow \frac{2}{\pi} \int_0^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx = \int_0^{\infty} e^{-x(a+b)} dx$$

$$= \left(\frac{e^{-(a+b)x}}{-(a+b)} \right)_0^{\infty} = \frac{1}{a+b} \left(0 - e^{-0} \right) = \frac{1}{a+b}$$

$$\therefore \int_0^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx = \frac{\pi}{2(a+b)}$$

$$\therefore \int_0^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx = \frac{\pi}{2(a+b)}$$